# The Electromagnetic Theory of Coaxial Transmission Lines and Cylindrical Shields

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A form of circuit which is of considerable interest for the transmission of high frequency currents is one consisting of a cylindrical conducting tube within which a smaller conductor is coaxially placed. Such tubes have found application in radio stations to connect transmitting and receiving apparatus to antennæ. As a part of the development work on such coaxial systems, it has been necessary to formulate the theory of transmission over a coaxial circuit and of the shielding against inductive effects which is afforded by the outer conductor. This paper deals generally with the transmission theory of coaxial circuits and extends the theory beyond the range of present application both as regards structure and frequency.

THE mathematical theory of wave propagation along a conductor with an external coaxial return is very old, going back to the work of Rayleigh, Heaviside and J. J. Thomson. Much important work has been done in developing and extending this theory. Among the problems dealt with in this development may be listed the following: the extension of the theory to systems consisting of a plurality of cylindrical conductors; the investigation of shielding and crosstalk in coaxial systems and the effects of eccentricity; the extension of the particular solution to include the complementary modes of propagation, etc.; and in general the adaptation of the mathematical theory to engineering uses, and its translation into the concepts and language of electric circuit theory. In addition to the author's contribution a substantial part of this mathematical work has been done by the group of engineers associated with Mr. John R. Carson, formerly of the American Telephone and Telegraph Company, now of the Bell Telephone Laboratories, Inc.

The problem is ideally adapted to mathematical investigation, because the conductor shape fits perfectly into the cylindrical system of coordinates, thereby making it entirely feasible to carry out a rigorous discussion on the basis of the electromagnetic theory, instead of using ordinary circuit theory. This has obvious advantages at ultra high frequencies, where the uncertainties of the circuit theory are conspicuous and not easily compensated for. It also proves to be of greater advantage at lower frequencies than one might at first assume. Fortunately, it turns out that the final results obtained by means of field theory can be expressed in a familiar language of circuit

theory, thereby gaining all the simplicity of the latter combined with all the accuracy of the former.

## Circularly Symmetric Electromagnetic Fields

In polar coordinates, Maxwell's equations assume the following form:

$$
\begin{aligned}
&\frac{\partial H_z}{\rho\partial\varphi} - \frac{\partial H_\varphi}{\partial z} = (g + i\omega\epsilon)E_\rho, \qquad \frac{\partial E_z}{\rho\partial\varphi} - \frac{\partial E_\varphi}{\partial z} = -\, i\omega\mu H_\rho, \\
&\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = (g + i\omega\epsilon)E_\varphi, \qquad \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -\, i\omega\mu H_\varphi, \\
&\frac{1}{\rho}\left[\frac{\partial(\rho H_\varphi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\varphi}\right] = (g + i\omega\epsilon)E_z, \\
&\frac{1}{\rho}\left[\frac{\partial(\rho E_\varphi)}{\partial\rho} - \frac{\partial E_\rho}{\partial\varphi}\right] = -\, i\omega\mu H_z,
\end{aligned}
\tag{1}
$$

where $E$ and $H$ are respectively the electromotive and magnetomotive intensities.[1]

In general, all six field components depend upon each other. If, however, these quantities are independent of either $\varphi$ or $z$, the partial derivatives with respect to the corresponding variable vanish and the original set of equations breaks up into two independent subsets, each involving only three physical quantities. Each of these special fields has important practical applications.

In the *circularly symmetric* case, that is, when the quantities are independent of $\varphi$, one of the independent subsets is composed of the first and the third equations on the left of (1), together with the second on the right:

$$
\begin{aligned}
&\frac{\partial(\rho H_\varphi)}{\partial\rho} = (g + i\omega\epsilon)\rho E_z, \qquad \frac{\partial H_\varphi}{\partial z} = -\,(g + i\omega\epsilon)E_\rho, \\
&\frac{\partial E_z}{\partial\rho} - \frac{\partial E_\rho}{\partial z} = i\omega\mu H_\varphi.
\end{aligned}
\tag{2}
$$

This *circular magnetic* field, with its lines of magnetomotive intensity

[1] In this paper we have adopted a unified practical system of units based upon the customary cgs system augmented by adding one typically electric unit. This system has three obvious advantages: first, theoretical results are expressed directly in the units habitually employed in the laboratory; second, the dimensional character and physical significance of such quantities as $i\omega\mu$ and $g + i\omega\epsilon$ are not obscured as in other systems by suppressing dimensions of some electrical unit such as permeability or dielectric constant; and third, the form of electromagnetic equations is very simple. In this system of units the electromotive intensity $E$ is measured in volts/cm., the magnetomotive intensity $H$ in amperes/cm., the intrinsic conductance $g$ in mhos/cm., the intrinsic inductance $\mu$ in henries/cm., and the intrinsic capacity $\epsilon$ in farads/cm. Thus, in empty space $\mu = 4\pi 10^{-9}$ henries/cm. or approximately 0.01257 $\mu h$/cm. and $\epsilon = (1/36\pi)\cdot 10^{-11}$ farads/cm. or approximately 0.0884 mmf./cm.

forming a system of coaxial circles, is associated with currents flowing in isolated wires as, for example, in a single vertical antenna and under ordinary operating conditions it is also found between the conductors of a coaxial pair (Fig. 1).

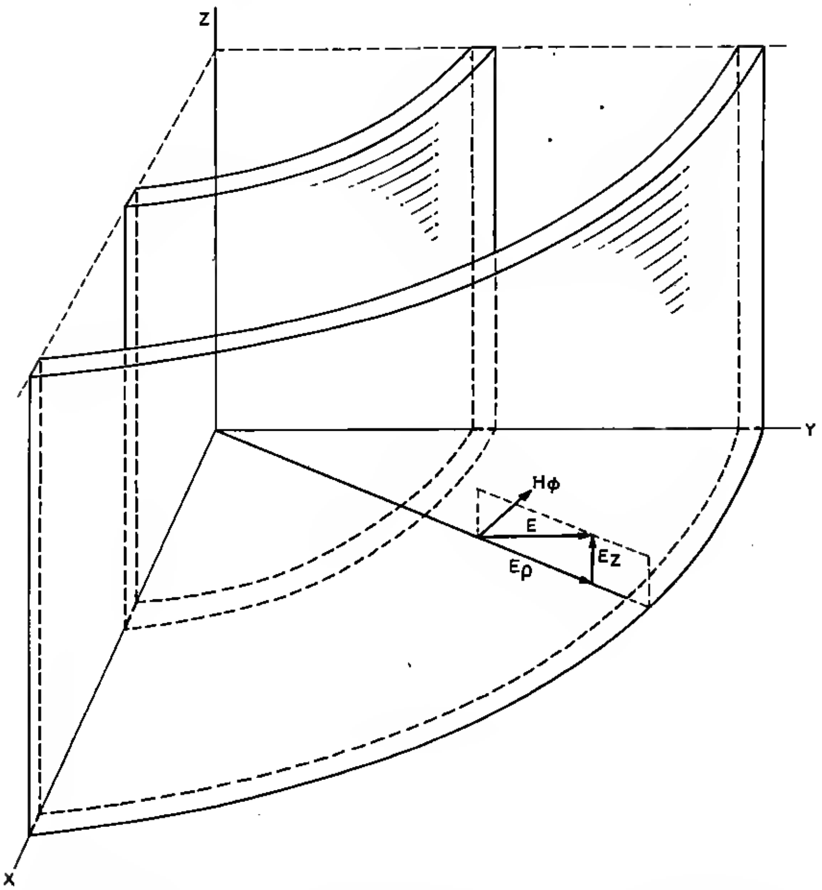


Fig. 1—The relative directions of the field components in a coaxial transmission line.

The remaining three equations of the set (1) form the second group:

$$\frac{\partial(\rho E_\varphi)}{\partial \rho} = -\,i\omega\mu\rho H_z, \qquad \frac{\partial E_\varphi}{\partial z} = i\omega\mu H_\rho,$$
$$(g + i\omega\epsilon)E_\varphi = \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}, \tag{3}$$

describing the *circular electric* field. Uniformly distributed electric

current in a circular turn of wire is surrounded by a field of this type; in this case, the lines of electromotive intensity form a coaxial system of circles.

## Two-Dimensional Fields

By definition, two-dimensional fields are constant in some one direction. If we take the $z$-axis of our reference system in this direction, all the partial derivatives with respect to $z$ vanish, $z$ disappears from our equations and we can confine our attention to any plane normal to the $z$-axis.

Once more the set of six electromagnetic equations breaks up into two independent subsets. One of these is [2]

$$E_\rho = -\frac{1}{(g+i\omega\epsilon)\rho}\frac{\partial H_z}{\partial\varphi}, \qquad E_\varphi = -\frac{1}{g+i\omega\epsilon}\frac{\partial H_z}{\partial\rho},$$
$$\frac{1}{\rho}\left[\frac{\partial(\rho E_\varphi)}{\partial\rho}+\frac{\partial E_\rho}{\partial\varphi}\right] = -\,i\omega\mu H_z. \tag{4}$$

The calculation of what is commonly known as "electrostatic" crosstalk between pairs of parallel wires is based upon these equations. For this reason we shall name the field defined by (4) the *electric field.*

Similarly, the remaining three equations define the magnetic field:

$$H_\rho = -\frac{1}{i\omega\mu\rho}\frac{\partial E_z}{\partial\varphi}, \qquad H_\varphi = -\frac{1}{i\omega\mu}\frac{\partial E_z}{\partial\rho},$$
$$\frac{1}{\rho}\left[\frac{\partial(\rho H_\varphi)}{\partial\rho}+\frac{\partial H_\rho}{\partial\varphi}\right] = -\,(g+i\omega\epsilon)E_z \tag{5}$$

and are useful in the theory of what is generally known as "electromagnetic" crosstalk.

The distinction between electric and magnetic fields is purely pragmatic and is based upon a necessary and valid engineering separation of general electromagnetic interference into two component parts. In some respects the firmly entrenched terms "electrostatic crosstalk" and "electromagnetic crosstalk" are unfortunate; it would be hopeless, however, to try a change of terminology at this late stage of engineering development.

Further consideration of two-dimensional fields will be deferred until the problem of shielding is taken up later in this paper (page 567).

[2] In passing from the original set (1) we reversed the sign of $E_\rho$ in order to make the set of equations symmetrical. The positive $E_\rho$ is now measured toward the axis.

[3] In these equations, the sign of $H_\varphi$ was reversed so that the magnetomotive intensity is now positive when it points clockwise. With this convention, the flow of energy is away from the axis when both $H_\varphi$ and $E_z$ are positive.

## Exponential Propagation

While electromotive forces could be applied in such a way that the fields would be of the kind given by (3), in the coaxial transmission line as actually energized the fields are of the circular magnetic type (2) which will claim our special attention in the next few sections.

In order to solve equations (2), we naturally want to eliminate all variables but one. This purpose can be readily accomplished if $E_z$ and $E_\rho$ are substituted from the first and the last equations of the set into the second. Thus, we obtain the following equation for the magnetomotive intensity:

$$\frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho}\right] + \frac{\partial^2 H_\varphi}{\partial z^2} = \sigma^2 H_\varphi, \tag{6}$$

where

$$\sigma^2 = g\omega\mu i - \omega^2\epsilon\mu. \tag{7}$$

Adopting the usual method of searching for particular solutions of (6) in the form

$$H_\varphi = R(\rho)Z(z), \tag{8}$$

where $R(\rho)$ is a function of $\rho$ alone, and $Z(z)$ a function of $z$ alone, we get

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \Gamma^2, \tag{9}$$

$$\frac{1}{R}\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d(\rho R)}{d\rho}\right] = \sigma^2 - \Gamma^2, \tag{10}$$

where $\Gamma$ is some constant about which we have no information for the time being.

Equation (9) is well known in transmission line theory; its general solution can be written in the form

$$Z = Ae^{\Gamma z} + Be^{-\Gamma z}, \tag{11}$$

where $A$ and $B$ are arbitrary constants. The solutions of (10) are Bessel functions. Since equation (6) is linear, we may invoke the principle of superposition and add any number of particular solutions corresponding to different values of $\Gamma$. Thus we can form an infinite variety of other solutions so as to satisfy the physical conditions of various practical problems.

It is seen at once from the first and the last equations of the set (2) that to each $H_\varphi$ of the form (8) there correspond an $E_z$ and $E_\rho$ of the same form; i.e., there exist circularly symmetric electromagnetic

fields, all of whose components vary exponentially in the direction of the axis of symmetry. Whether any of these fields can be produced individually by some simple physical means is impossible to decide on theoretical grounds alone. It may happen, of course, that the field due to *any* practically realizable source is always a combination of several simple exponential fields. In any case, however, we want to know the properties of pure exponential solutions.

It is convenient to make the exponential character of the quantities $E_\rho$, $E_z$ and $H_\varphi$ explicit and write them respectively in the form $E_\rho e^{-\Gamma z}$, $E_z e^{-\Gamma z}$ and $H_\varphi e^{-\Gamma z}$. The new quantities $E_\rho$, $E_z$ and $H_\varphi$ are functions of $\rho$ only. If the suggested substitution is made in equations (2), the factor $e^{-\Gamma z}$ cancels out and we have

$$E_\rho = \frac{\Gamma}{g + i\omega\epsilon} H_\varphi, \qquad i\omega\mu H_\varphi = \frac{dE_z}{d\rho} + \Gamma E_\rho, \qquad \frac{d(\rho H_\varphi)}{d\rho} = (g + i\omega\epsilon)\rho E_z. \tag{12}$$

The quantity $\Gamma$ is called the *longitudinal propagation constant* or simply the *propagation constant* when no confusion is possible.[4]

Recalling the implied exponential time factor $e^{i\omega t}$, we see that the complete exponential factor in the expressions for the field intensities is $e^{-\Gamma z + i\omega t}$. The propagation constant $\Gamma$ is often a complex number and can be represented in the form $\alpha + i\beta$ where the real part is called the *attenuation constant* and the imaginary part, the *phase constant*. Thus, $e^{-\alpha z}$ measures the decrease in the amplitudes of the intensities and $e^{-i(\beta z - \omega t)}$, the change of their phases in time as well as in the $z$-direction. The latter factor suggests that we are dealing with a wave moving in the positive direction of the $z$-axis with a velocity $(\omega/\beta)$. A wave moving in the opposite direction is obtained by reversing the sign of $\Gamma$.

## Perfectly Conducting Coaxial Cylinders [5]

Let us now consider one of the simplest problems which, though purely academic in itself, will throw some light on what is likely to happen under less ideal conditions. We suppose that a perfect dielectric is enclosed between two perfectly conducting coaxial cylinders (Fig. 1) whose radii [6] are $b$ and $a$ $(b < a)$. Our problem is to find the symmetric electromagnetic fields which can exist in such a medium.

[4] Another set of exponential solutions is obtained from this by changing $\Gamma$ into $-\Gamma$.

[5] For a thorough discussion of "complementary" waves in coaxial pairs the reader is referred to John R. Carson [4].

[6] Only the outer radius of the inner conductor and the inner radius of the outer conductor need be considered because in perfectly conducting media electric states are entirely surface phenomena.

In a perfect dielectric $g = 0$ and the preceding set of equations becomes

$$E_\rho = \frac{\Gamma}{i\omega\epsilon} H_\varphi, \qquad i\omega\mu H_\varphi = \frac{dE_z}{d\rho} + \Gamma E_\rho, \tag{13}$$
$$\frac{d(\rho H_\varphi)}{d\rho} = i\omega\epsilon\rho E_z.$$

No force is required to sustain electric current in perfect conductors and the tangential components of the intensities are continuous across the boundaries between different media; therefore, the longitudinal electromotive intensity vanishes where $\rho$ equals either $a$ or $b$.

Substituting $E_\rho$ from the first equation into the second, solving the latter for $H_\varphi$ and inserting it into the third equation, we have successively

$$H_\varphi = -\frac{i\omega\epsilon}{m^2}\frac{dE_z}{d\rho}, \tag{14}$$

and

$$\rho\frac{d^2E_z}{d\rho^2} + \frac{dE_z}{d\rho} + m^2\rho E_z = 0, \tag{15}$$

where, for convenience, we let $\Gamma^2 + \omega^2\epsilon\mu = m^2$. The most general solution of the last equation is usually written in the form

$$E_z(\rho) = AJ_0(m\rho) + BY_0(m\rho), \tag{16}$$

where $J_0$ and $Y_0$ are Bessel functions of order zero and $A$ and $B$ are constants so far unknown.[7]

The constants $A$ and $B$ can be determined from the fact already mentioned that $E_z$ vanishes on the surface of either conductor, i.e., from the following equations:

$$AJ_0(mb) + BY_0(mb) = 0,$$

and

$$AJ_0(ma) + BY_0(ma) = 0. \tag{17}$$

These equations are certainly satisfied if both constants are equal to 0. If, however, they are not equal to 0 simultaneously, we can determine their ratio from each equation of the above system. These ratios should be the same, of course, and yet they cannot be equal for every value of $m$. Thus, the *permissible* values of $m$ are the roots of

[7] For large values of the argument these Bessel functions are very much like slightly damped sinusoidal functions; in fact $J_0(x)$ and $Y_0(x)$ are approximately equal, respectively, to $\sqrt{2/\pi x}\cos(x - \pi/4)$ and $\sqrt{2/\pi x}\sin(x - \pi/4)$, provided $x$ is large enough.

the following equation:

$$-\frac{A}{B} = \frac{Y_0(mb)}{J_0(mb)} = \frac{Y_0(ma)}{J_0(ma)} \,. \tag{18}$$

This equation has an infinite number of roots [8] whose approximate values can be readily determined if we replace Bessel functions by their approximations in terms of circular functions. Thus, we have

$$m_n = \frac{n\pi}{a - b}\,, \qquad (n = 1, 2, 3, \cdots). \tag{19}$$

This is a surprisingly good approximation for *all* roots if the radius of the outer conductor is less than three times that of the inner; and the larger the $n$, the better the approximation.[9] The propagation constants are computed from the corresponding values of $m_n$ by means of the following equation,

$$\Gamma_n = \sqrt{m_n^2 - \omega^2\epsilon\mu}. \tag{20}$$

First of all, let us study the simplest solution in which both $A$ and $B$ vanish. In this case, the longitudinal electromotive intensity vanishes identically. The magnetomotive intensity—and the transverse electromotive intensity, as well—also vanishes unless the denominator $m^2$ in equation (14) equals zero. If all intensities were to vanish, we should have no field and there would be nothing to talk about; hence, we take the other alternative and let

$$\Gamma^2 + \omega^2\epsilon\mu = 0, \qquad \text{i.e.,} \qquad \Gamma = i\omega\sqrt{\epsilon\mu}, \tag{21}$$

the positive sign having been implied in writing equations (13). In air, $\epsilon\mu = (1/c^2)$ where $c$ is the velocity of light in cm.; hence, in air this particular propagation constant equals $i\omega/c$. Since $E_z$ equals zero everywhere, the electromotive intensity is wholly transverse; and the flow of energy being, according to Poynting, at right angles to the electromotive and magnetomotive intensities, the energy transfer is wholly longitudinal.

The above method of determining the propagation constant may be open to suspicion; besides, the method does not tell how to obtain the actual values of the electromagnetic intensities but merely leads to a relation compatible with the existence of such intensities. Therefore, let us obtain the wanted information directly from the funda-

[8] A. Gray and G. B. Mathews, "A Treatise on Bessel Functions" (1922), p. 261.

[9] It is strictly accurate if the radii of the cylinders are infinite, i.e., if we are dealing with a dielectric slab bounded by perfectly conducting planes.

mental equations (13) which assume the following simple form:

$$i\omega\epsilon E_\rho = \Gamma H_\varphi, \qquad i\omega\mu H_\varphi = \Gamma E_\rho, \qquad \frac{d(\rho H_\varphi)}{d\rho} = 0, \tag{22}$$

if $E_z$ vanishes identically. Either of the first two equations determines the ratio of the electromotive intensity to the magnetomotive; the two ratios are consistent only if the condition (21) is satisfied. Then, we have also

$$E_\rho = \frac{\Gamma}{i\omega\epsilon} H_\varphi = \sqrt{\frac{\mu}{\epsilon}}\, H_\varphi \qquad \text{and} \qquad H_\varphi = \frac{A}{\rho}, \tag{23}$$

where $A$ is some quantity independent of $\rho$. This constant can be readily calculated from Ampere's law. The magnetomotive force acting along the circumference of any particular cross-section of the inner cylinder equals $2\pi\rho H_\varphi$ amperes, i.e., $2\pi A$; since this M.M.F. should equal the total current $I$ flowing in the inner conductor through the cross-section, the quantity $A$ equals $I/2\pi$. Reintroducing the implied factor $e^{-\Gamma z}$, we have

$$\begin{aligned} H_\varphi &= \frac{I}{2\pi\rho}\, e^{-\Gamma z}, \\ E_\rho &= \frac{I}{2\pi\rho}\sqrt{\frac{\mu}{\epsilon}}\, e^{-\Gamma z}. \end{aligned} \tag{24}$$

To here

In practical measurements we are concerned with the total potential difference ($V$) between the cylinders, rather than with the transverse electromotive intensity. The former is merely the integral of the intensity,

$$V = \int_b^a E_\rho d\rho = \left(\frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \log\frac{a}{b}\right) I e^{-\Gamma z}. \tag{25}$$

*This voltage and the current $I$ vary as voltage and current in a semi-infinite transmission line whose propagation constant is $\Gamma$ and whose characteristic impedance is*

$$Z_0 = \frac{V}{Ie^{-\Gamma z}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \log\frac{a}{b}\,. \tag{26}$$

*At any point $z$ the intensities $E_\rho$ and $H_\varphi$ have the same values as would the voltage and current at the same distance $z$ from the end of a transmission line whose propagation constant and characteristic impedance are respectively $i\omega\sqrt{\epsilon\mu}$ and $\sqrt{\mu/\epsilon}$.*

The connection between electromagnetic theory and line theory is so important that, risking repetition, we wish to emphasize their intimate relationship by deriving the well-known differential equations of the line theory directly from the electromagnetic equations (2) *combined with the assumption that the longitudinal electromotive intensity vanishes everywhere.* We already know that under the assumed conditions the first equation of the system (2) becomes

$$H_\varphi = \frac{I}{2\pi\rho}, \tag{27}$$

where $I$ is the total current flowing in the inner cylinder through a particular cross-section and is some function [10] of $z$. We can therefore rewrite the last two equations of the system as follows:

$$\frac{\partial E_\rho}{\partial z} = -\frac{i\omega\mu}{2\pi\rho} I, \qquad \frac{1}{2\pi\rho}\frac{\partial I}{\partial z} = -i\omega\epsilon E_\rho. \tag{28}$$

We have merely to integrate both equations with respect to $\rho$ from $b$ to $a$ and substitute the potential difference $V$ for the integral of the transverse electromotive intensity to obtain

$$\frac{\partial V}{\partial z} = -\left(\frac{i\omega\mu}{2\pi}\log\frac{a}{b}\right) I, \qquad \frac{\partial I}{\partial z} = -\frac{2\pi i\omega\epsilon}{\log\frac{a}{b}} V, \tag{29}$$

which are the equations of the transmission line whose distributed series inductance equals $(\mu/2\pi)\log(a/b)$ henries/cm. and shunt capacity $2\pi\epsilon/(\log a/b)$ farads/cm.

With this, we conclude the special case in which the longitudinal electromotive intensity vanishes everywhere, the propagation constant equals $i\omega\sqrt{\epsilon\mu}$, and the velocity of transmission is that of light.

We now turn our attention to the case in which $A$ and $B$ do not vanish. We have already noted that the propagation constants are given by equation (20). Since, in this case, we are interested primarily in the nature of the phenomena rather than in the details of field distribution, we shall simplify our mathematics by supposing the radii of the cylinders to be infinite. Thus, the cylinders become two planes perpendicular to the $x$-axis, distance $a$ apart. The $\varphi$-direction, then, coincides with the $y$-direction and, therefore, all the intensities are independent of the $y$-coordinate. Let us choose the $z$-axis halfway between the planes. The equations describing this two-dimen-

[10] On this occasion, we should remember that a particular type of this function had not yet been ascertained at the time the equations (2) were arrived at.

sional transmission line are

$$
\begin{aligned}
\frac{\partial H_y}{\partial x} &= i\omega\epsilon E_z, \\
\frac{\partial H_y}{\partial z} &= -\, i\omega\epsilon E_x, \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\, i\omega\mu H_y.
\end{aligned}
\tag{30}
$$

If $n$ is an odd integer, these possess the following solutions:

$$
\begin{aligned}
E_x &= A\frac{\Gamma_n}{i\omega\epsilon}\sin\frac{n\pi x}{a}, \\
E_z &= A\frac{n\pi}{i\omega\epsilon a}\cos\frac{n\pi x}{a}, \\
H_y &= A\sin\frac{n\pi x}{a};
\end{aligned}
\tag{31}
$$

and if $n$ is an even integer,

$$
\begin{aligned}
E_x &= A\frac{\Gamma_n}{i\omega\epsilon}\cos\frac{n\pi x}{a}, \\
E_z &= -\, A\frac{n\pi}{i\omega\epsilon a}\sin\frac{n\pi x}{a}, \\
H_y &= A\cos\frac{n\pi x}{a},
\end{aligned}
\tag{32}
$$

where

$$
\Gamma_n = \sqrt{\frac{n^2\pi^2}{a^2} - \omega^2\epsilon\mu} = \pi\sqrt{\frac{n^2}{a^2} - \frac{4}{\lambda^2}}, \tag{33}
$$

and $\lambda$ is the wave-length corresponding to the frequency $f$.

Let us now define the *longitudinal impedance* ($Z_z$) as the ratio of $E_x$ to $H_y$,

$$
Z_z = \frac{\Gamma_n}{i\omega\epsilon}, \tag{34}
$$

and the *transverse impedance* (the impedance in the $x$-direction) as the ratio of $E_z$ to $H_y$,

$$
\begin{aligned}
Z_x &= \frac{n\pi}{i\omega\epsilon a}\cot\frac{n\pi x}{a}, && \text{if } n \text{ is odd,} \\
Z_x &= -\,\frac{n\pi}{i\omega\epsilon a}\tan\frac{n\pi x}{a}, && \text{if } n \text{ is even.}
\end{aligned}
\tag{35}
$$

It will be observed that, depending on the frequency, the longitudinal

propagation constant $\Gamma_n$ is either real or purely imaginary; it vanishes if $a = n(\lambda/2)$, that is, if the spacing between the planes is a whole number of half wave-lengths. When the propagation constant is real, the longitudinal impedance is purely imaginary, and vice versa, when the propagation constant is purely imaginary, the longitudinal impedance is real. In the former case, no energy is transmitted longitudinally but merely surges back and forth, and in the latter case we have a true transmission line. The transverse impedance is purely imaginary at all frequencies and, hence, the energy merely fluctuates to and fro.

If the frequency is sufficiently low, all of these higher order propagation constants are real and all the energy is transmitted in the *principal mode* described by equations (21) to (29). The rôle of the higher propagation constants consists in redistributing the energy near the sending terminal,[11] that is; in terminal distortion. But as the frequency gets high enough to make the wave-length less than $2a$, the next transmission mode may become prominent, and so forth up the infinite ladder of transmission modes.

## Imperfect Coaxial Conductors [12]

We shall now suppose that the conductors are not perfect; i.e., the conductivity instead of being infinite, is merely large. Assuming that our solutions are continuous functions of conductivity (this can be proved), we conclude: first, there exists an infinite series of propagation constants approaching the values given in the preceding section as the conductivity tends to infinity; second, one of these propagation constants, namely that approaching $i\omega\sqrt{\epsilon\mu}$, is very small unless the conductivity is too small. In the immediately succeeding sections we shall be concerned only with electromagnetic fields corresponding to this particular propagation constant.

Let us now prove that the simple expression for the magnetomotive intensity in the dielectric between perfectly conducting cylinders is still true for all practical purposes, even if the conductors are merely good, and even when there are more than two of them. Since the lines of force are circles, coaxial with the conductors, and since $H_\varphi$ is independent of $\varphi$, the total magnetomotive force acting along any one of the circles equals $H_\varphi$ times the circumference of the circle $(2\pi\rho)$. This M.M.F. also equals the total current $I$ passing through the area of the circle. Therefore, the magnetomotive intensity is $(I/2\pi\rho)$ amperes/cm. This expression is true at any point in the conductors as

[11] And near the receiving terminal as well, if the line is finite.

[12] The general theory of wave propagation in a multiple system of imperfect coaxial conductors is amply covered by John R. Carson and J. J. Gilbert [2, 3].

well as in the dielectric between them. In a conductor the total current $I$ passing through the area of the circle is a function of $\rho$ since the current is distributed throughout the entire cross-section of the conductor. Strictly speaking, the same is true of any circle in the dielectric. There is one important difference, however; the conduction current passing through such circles is the same and the displacement current is usually so small that it can be legitimately neglected. Thus, in the dielectric, we have to an extremely high degree of accuracy unless $\rho$ is very large

$$H_\varphi = \frac{I}{2\pi\rho}, \tag{36}$$

where $I$ is merely a constant, namely, the total *conduction* current passing through the area of the circle of radius $\rho$.

That the longitudinal displacement current can be neglected, unless the conductivity of the conductors is small, has been already indicated in the opening paragraph. The following comparison is an aid to the mathematical argument. The density of the longitudinal conduction current is $gE$ and that of the displacement current is $i\omega\epsilon E$. Near the boundary, $E$ is substantially the same in the conductor and in the dielectric. In copper, $g = (1/1.724)10^6$ and in air $\epsilon = (1/36\pi)10^{-11}$. Thus, even at very high frequencies, the density of the displacement current is very small compared to that of the conduction current. On the other hand, the conduction current is ordinarily distributed over a small area while the displacement current may flow across a large area. The latter area would have to be very large, however, before it could even begin to compensate for the extremely low current density.

## Electromotive Intensities in Dielectrics

With the aid of equations (12) and (36), we can now calculate the electromotive intensities in the dielectric between two conductors. Thus, the transverse intensity is

$$E_\rho = \frac{\Gamma I}{2\pi(g + i\omega\epsilon)\rho}. \tag{37}$$

Substituting this in the second equation of the set (12), we obtain the following differential equation for the longitudinal intensity:

$$\frac{dE_z}{d\rho} = \left[ i\omega\mu - \frac{\Gamma^2}{g + i\omega\epsilon} \right] \frac{I}{2\pi\rho}, \tag{38}$$

where $\mu$ is the permeability of the dielectric. Integrating with respect

to $\rho$, we have

$$E_z = \frac{1}{2\pi}\left[ i\omega\mu - \frac{\Gamma^2}{g + i\omega\epsilon} \right] I \log \frac{\rho}{b'} + A, \tag{39}$$

where $A$ is a constant to be determined from the boundary conditions.[13]

THE POTENTIAL DIFFERENCE BETWEEN TWO COAXIAL CYLINDERS

Equation (36) relates the transverse electromotive intensity to the total current flowing in the inner conductor. In practice, however, we are interested in the difference of potential between the conductors, that is, in the transverse electromotive force rather than the electromotive intensity. This potential difference $V$ is obtained at once from equation (37) by integration:

$$V = \int_{b'}^{a''} E_\rho d\rho = \frac{\Gamma I}{2\pi(g + i\omega\epsilon)} \int_{b'}^{a''} \frac{d\rho}{\rho} = \frac{\Gamma I \log \dfrac{a''}{b'}}{2\pi(g + i\omega\epsilon)}. \tag{40}$$

This transverse E.M.F. produces a transverse electric current which is partly a conduction current—if the dielectric is not quite perfect—and partly a displacement (or "capacity") current.

Now, the total transverse current per centimeter length of line is

$$I_\rho = 2\pi\rho(g + i\omega\epsilon)E_\rho.$$

Then, by equation (37), we have

$$I_\rho = \Gamma I. \tag{41}$$

Therefore, equation (40) becomes

$$V = \frac{\log \dfrac{a''}{b'}}{2\pi(g + i\omega\epsilon)} I_\rho. \tag{42}$$

The ratio of a current to the electromotive force that produces it is called *admittance.* Hence, the distributed *radial admittance* per

[13] The following system of notation will be adhered to throughout the remainder of the paper: The inner radius of any cylindrical conductor is denoted by $a$, and its outer radius by $b$. When several coaxial conductors are used, they are differentiated by superscripts; $a'$, $a''$, $a^{(3)}$, $\cdots$ referring to their inner radii, for example, and $b'$, $b''$, $b^{(3)}$, $\cdots$ to their outer radii. This convention also applies to conductivities, permeabilities, and other physical constants of the conductors in question.

For convenience, we have written the ratio of $\rho$ to the outer radius of the inner conductor in place of $\rho$; this change affects only the arbitrary constant $A$ which will eventually be assigned the value required by the boundary conditions. When written in this form, the first term of $E_z$ vanishes on the surface of the inner conductor which is a convenience in determining the value of $A$.

unit length between two cylindrical conductors is

$$Y = \frac{2\pi(g + i\omega\epsilon)}{\log \frac{a''}{b'}} \equiv G + i\omega C, \tag{43}$$

the symbols $G$ and $C$ being used in the usual way to designate the distributed radial conductance and capacity. Writing these separately, we have

$$G = \frac{2\pi g}{\log \frac{a''}{b'}}, \qquad C = \frac{2\pi\epsilon}{\log \frac{a''}{b'}}. \tag{44}$$

Returning to (40), we find that $V$ can be written in the form

$$V = \frac{\Gamma}{Y} I.$$

But the ratio of the transverse electromotive force $V$ to the longitudinal current $I$ is known as the longitudinal *characteristic impedance* of the coaxial pair. Its value is obviously $\Gamma / Y$.

## The External Inductance

In dealing with parallel wires it is customary to use the term "external inductance" for the total magnetic flux in the space surrounding the pair.[14] We shall adopt the same usage in connection with coaxial pairs. Strictly speaking, we must therefore consider it as being composed of two parts: one being the flux *between* the cylinders, the other the flux in the space surrounding them. But the longitudinal displacement current is negligible by comparison with the conduction current, and effects due to it have been consistently ignored throughout this part of our study. To the same order of approximation, the flux outside the pair is negligible by comparison with that between them, whence we find the "external inductance" to be

$$L_e = \frac{\mu \int_{b'}^{a''} H_\varphi d\rho}{I} = \frac{\mu}{2\pi} \log \frac{a''}{b'} \text{ henries/cm.} \tag{45}$$

[14] While this definition is very descriptive, it is not strictly accurate unless the wires are perfectly conducting. The correct definition should read as follows: The external inductance of a parallel pair is the measure (per unit current) of magnetic energy stored in the space surrounding the pair. The reason the simpler definition fails for imperfectly conducting parallel wires is because some of the lines of magnetic flux lie partly inside and partly outside the wires. This does not happen in connection with coaxial pairs even when they are not perfectly conducting. Hence we are warranted in using the simpler idea.

Comparing this with equation (44), we have the following relation between the external inductance and the capacity

$$CL_e = \epsilon\mu. \tag{46}$$

### Propagation Constants of Coaxial Pairs

Since the relation between electromotive intensity and current is linear, we are justified in writing the intensities at the adjacent surfaces of the pair in the form

$$E_z(b') = Z_b'I, \qquad E_z(a'') = Z_a''I, \tag{47}$$

where $Z_b'$ and $Z_a''$ depend only upon the material of the conductors and the geometry of the system. These quantities will be called *surface impedances* of the inner and outer conductors, respectively.

Inserting (47) in (39) we obtain

$$\begin{aligned} A &= Z_b'I, \\ \frac{1}{2\pi}\left[i\omega\mu - \frac{\Gamma^2}{g + i\omega\epsilon}\right] I \log\frac{a''}{b'} + A &= -Z_a''I, \end{aligned} \tag{48}$$

by means of which $A$ and $\Gamma$ may be expressed in terms of $Z_b'$ and $Z_a''$. If we solve the first of these for $A$ and substitute the value thus derived in the second we get, by virtue of (45),

$$\frac{\Gamma^2}{2\pi(g + i\omega\epsilon)} \log\frac{a''}{b'} = Z_a'' + Z_b'' + i\omega L_e, \tag{49}$$

or, by (43)

$$\Gamma^2 = YZ, \tag{50}$$

where for brevity we have written

$$Z = Z_a'' + Z_b' + i\omega L_e. \tag{51}$$

### Direct Conversion of the Circularly Symmetric Field Equations into Transmission Line Equations

As the practical applications of Maxwell's theory become more numerous, it becomes increasingly important to formulate its exact connection with transmission line theory. With this purpose in mind, let us attempt to throw (2) into the form of the transmission line equations.

The obvious plan of attack is to introduce into (2) the transverse voltage $V$ and the longitudinal current $I$, in place of the intensities $E$ and $H$. The total current is introduced by substituting $(I/2\pi\rho)$ for $H_\varphi$, and the total voltage by integrating the set of equations (2) in the transverse direction. The first equation gives us nothing of

importance.[15] The second and third equations, on the other hand, give

$$\frac{i\omega\mu I}{2\pi}\log\frac{a''}{b'} = E_z''(a) - E_z'(b) - \frac{\partial V}{\partial z}, \tag{52}$$

$$V = -\frac{\log\frac{a''}{b'}}{2\pi(g+i\omega\epsilon)}\frac{\partial I}{\partial z}.$$

But, upon substituting (45), (47) and (51) in the first of these equations and (43) in the second, we get

$$\frac{\partial V}{\partial z} = -ZI, \qquad \frac{\partial I}{\partial z} = -YV, \tag{53}$$

where $Z$ and $Y$ are to be interpreted respectively as the distributed series impedance and shunt admittance.

## Current Distribution in Cylindrical Conductors

So far, we have been dealing with electromagnetic intensities in dielectrics. We now turn our attention to conductors and determine their current distributions with the ultimate view of calculating their surface impedances. One of our sources of information is the familiar set of equations (12). In these equations, however, we now let $\epsilon = 0$ since the displacement current in conductors is negligibly small by comparison with the conduction current. From these equations, we eliminate electromotive intensities and thus obtain a differential equation for the magnetomotive intensity. The latter is in fact equation (6) with only one difference: the exponential factor $e^{-\Gamma z}$ has been explicitly introduced and cancelled so that the equation has become

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d(\rho H_\varphi)}{d\rho}\right] = (\sigma^2 - \Gamma^2)H_\varphi,$$

or

$$\frac{d^2H_\varphi}{d\rho^2} + \frac{1}{\rho}\frac{dH_\varphi}{d\rho} - \frac{H_\varphi}{\rho^2} = (\sigma^2 - \Gamma^2)H_\varphi, \tag{54}$$

where

$$\sigma^2 = g\omega\mu i = 2\pi g\mu f i.$$

This $\sigma$ will be called the intrinsic propagation constant of solid metal.

[15] Our standard practice of neglecting the longitudinal displacement currents has given us the general rule that $2\pi\rho H_\varphi = I$ is independent of $\rho$. Using this relation in the first of equations (2.2), we get

$$(g + i\omega\epsilon)E_z \doteq 0;$$

but this merely reflects the fact that $g + i\omega\epsilon$ is very small.

The attenuation and the phase constants are each equal to $\sqrt{\pi g\mu f}$. The intrinsic propagation constants of metals are large quantities except at low frequencies as the accompanying table indicates.

PROPAGATION CONSTANT OF COMMERCIAL COPPER

$g = 5.800\ 10^5$ mhos/cm.
$\mu = 0.01257\ \mu h$/cm.

| $f$ | $\frac{\sigma}{1+i} = \sqrt{\pi g\mu f}$ |
|---|---|
| 0 | 0.0 |
| 1 | 0.1513 |
| 10 | 0.4785 |
| 100 | 1.513 |
| 10,000 | 15.13 |
| 1,000,000 | 151.3 |
| 100,000,000 | 1513. |

On the other hand, $\Gamma$ is very small; if air is the dielectric between the conductors, $\Gamma$ is of the order of $(1/3)i\omega\ 10^{-10}$. Hence, even at high frequencies $\Gamma^2$ is negligibly small by comparison with $\sigma^2$ and we can rewrite (54) as follows:

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho H_\varphi)\right] = \sigma^2 H_\varphi. \tag{55}$$

This is Bessel's equation and its solution can be written down at once [16] as

$$H_\varphi = AI_1(\sigma\rho) + BK_1(\sigma\rho), \tag{56}$$

where the functions $I_1(u)$ and $K_1(u)$ are the *modified Bessel functions of the first order and respectively of the first and second kind.* For large values of the argument we have approximately

$$\begin{aligned} I_1(u) &= \frac{e^u}{\sqrt{2\pi u}}\left(1 - \frac{3}{8u}\right), \\ K_1(u) &= \sqrt{\frac{\pi}{2u}}\, e^{-u}\left(1 + \frac{3}{8u}\right); \end{aligned} \tag{57}$$

[16] It is interesting to note that in the case of a fairly thin hollow conductor whose inner radius is not too small there exist very simple approximate solutions of (55). Under these circumstances $\rho$ varies over such a small range that no serious error is introduced in treating the factors $(1/\rho)$ and $\rho$ in (55) as constants, and the equation becomes

$$\frac{d^2H_\varphi}{d\rho^2} = \sigma^2 H_\varphi,$$

which is satisfied by the exponential functions $e^{\sigma\rho}$ and $e^{-\sigma\rho}$. The larger the value of $\rho$ and the faster the change in $H_\varphi$ with $\rho$, the better is the approximation.

while for small values

$$I_1(u) = \frac{u}{2} + \frac{u^3}{16},$$
$$K_1(u) = \frac{1}{u} + \frac{u}{2}\log\frac{u}{2}. \tag{58}$$

The function $I_1(u)$ becomes infinite and $K_1(u)$ vanishes when $u$ is infinite.[17] When $u$ is zero, $I_1(u)$ vanishes and $K_1(u)$ becomes infinite.

The longitudinal electromotive intensity is calculated from the third equation (12) with the aid of the following rules for differentiation of modified Bessel functions of any order $n$:

$$\frac{d}{dx}(x^n I_n) = x^n I_{n-1},$$
$$\frac{d}{dx}(x^n K_n) = -x^n K_{n-1}. \tag{59}$$

Thus,

$$E_z = \eta[AI_0(\sigma\rho) - BK_0(\sigma\rho)], \tag{60}$$

where

$$\eta = \frac{\sigma}{g} = \frac{i\omega\mu}{\sigma}. \tag{61}$$

For reasons which will appear later, this quantity $\eta$ will be called the *intrinsic impedance of solid metal.*

The current density is merely the product of the intensity $E_z$ and the conductivity $g$.

In a general way the behavior of the functions of zero order is similar to that of the functions whose order is unity. Thus, for large values of the argument,

$$I_0(u) = \frac{e^u}{\sqrt{2\pi u}}\left(1 + \frac{1}{8u}\right),$$
$$K_0(u) = \sqrt{\frac{\pi}{2u}}\,e^{-u}\left(1 - \frac{1}{8u}\right), \tag{62}$$

and for small values

$$I_0(u) = 1 - \frac{u^2}{4},$$
$$K_0(u) = -\log u + 0.116. \tag{63}$$

[17] This statement is correct only as long as the real part of $u$ is positive. This is so in our case because out of two possible values of the square root representing $\sigma$ we can always choose the one with the positive real part.

## Surface Impedance of a Solid Wire

On page 547 we defined the surface impedances of a coaxial pair as the ratios of the longitudinal electromotive intensities on the adjacent surfaces of the cylinders to the total currents flowing in the respective conductors. In that place, however, we were unable to give explicit formulæ for the impedances so defined because we did not yet have a precise value for $E_z$. Now that this omission has been supplied, we are prepared to compute $Z_b'$ and $Z_a''$.

We consider the case of a *solid* inner cylinder surrounded by *any coaxial return*, and seek to determine the constants $A$ and $B$ in (60). Since the E.M.I. must be finite along the axis of the wire we must make $B = 0$, because the $K$-function becomes infinite when $\rho = 0$. On the surface of the wire the magnetomotive intensity is $I/2\pi b$ if $I$ is the total current in the wire. By equation (56) this intensity equals $AI_1(\sigma b)$; hence,

$$A = \frac{I}{2\pi b I_1(\sigma b)}$$

and the final expression for the electromotive intensity within the wire is

$$E_z(\rho) = \frac{\eta I_0(\sigma\rho)}{2\pi b I_1(\sigma b)} I. \tag{64}$$

Thus, we have the following expression for the surface impedance of the solid wire:

$$Z_b = \frac{E_z(b)}{I} = \frac{\eta I_0(\sigma b)}{2\pi b I_1(\sigma b)}, \text{ ohms/cm.} \tag{65}$$

As the argument increases, the modified Bessel functions of the first kind (the $I$-functions) become more and more nearly proportional to the exponential functions of the same argument. Thus, if the absolute value of $\sigma b$ exceeds 50, the Bessel functions in the preceding equation cancel out and the following simple formula holds within 1 per cent:

$$Z_b = \frac{\eta}{2\pi b} = \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}\,(1 + i), \text{ ohms/cm.} \tag{66}$$

This surface impedance consists of a resistance representing the amount of energy dissipated in heat, and a reactance due to the magnetic flux in the wire itself. Separating (66) into these two parts, we have, approximately,

$$R_b = \omega L_b = \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}.$$

However, most of the error in (66) occurs in the real part. If more accurate approximations for Bessel functions are used, then

$$\begin{aligned} R_b &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}} + \frac{1}{4\pi g b^2}, \\ \omega L_b &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}; \end{aligned} \tag{67}$$

these are correct within 1 per cent if $|\sigma b| > 6$. The surface inductance $L_b$ equals $(1/4\pi b)\sqrt{\mu/\pi g f}$ henries/cm.; it decreases as the frequency increases.

If the wire is so thin or the frequency is so low that $|\sigma b| < 6$, equation (65) has to be used. Its use in computations is quite simple, however, because the argument $\sigma b$ is a complex number of the form $u\sqrt{i}$; and the necessary functions have been tabulated. Lord Kelvin introduced the symbols ber $u$ and bei $u$ for the real and the imaginary parts of $I_0(u\sqrt{i})$, so that we now write

$$I_0(u\sqrt{i}) = \operatorname{ber} u + i \operatorname{bei} u. \tag{68}$$

Differentiating, we have

$$\sqrt{i}\, I_0'(u\sqrt{i}) = \sqrt{i}\, I_1(u\sqrt{i}) = \operatorname{ber}' u + i \operatorname{bei}' u,$$

and therefore

$$I_1(u\sqrt{i}) = \frac{\operatorname{ber}' u + i \operatorname{bei}' u}{\sqrt{i}}. \tag{69}$$

If we insert these values in (65), and recall that the d.-c. resistance of a solid wire is $1/\pi g b^2$, and that $\sigma = g\eta$, we obtain at once

$$\begin{aligned} \frac{Z_b}{R_{\text{d.-c.}}} = \pi g b^2 Z_b = \frac{u}{2}\,\frac{\operatorname{ber} u \operatorname{bei}' u - \operatorname{bei} u \operatorname{ber}' u}{(\operatorname{ber}' u)^2 + (\operatorname{bei}' u)^2} \\ + i\,\frac{u}{2}\,\frac{\operatorname{ber} u \operatorname{ber}' u + \operatorname{bei} u \operatorname{bei}' u}{(\operatorname{ber}' u)^2 + (\operatorname{bei}' u)^2}, \end{aligned} \tag{70}$$

where $u$ is the absolute value of $\sigma b$. The accompanying graph illustrates the real and imaginary parts of this equation [18] (Fig. 2).

### The Surface Impedances of Hollow Cylindrical Shells [19]

In the case of a hollow conductor whose inner and outer radii are respectively equal to $a$ and $b$, the return coaxial path for the current

[18] For equation (70) and various approximations see E. Jahnke and F. Emde.

[19] In the case of self-impedances the more general equations of two parallel cylindrical shells were deduced by Mrs. S. P. Mead. For the special formulæ concerning self-impedances of coaxial pairs see A. Russell.

may be provided either outside the given conductor or inside it or partly inside and partly outside. We designate by $Z_{aa}$ the surface impedance with internal return, and by $Z_{bb}$, that with external return. These impedances are equal only at zero frequency; but if the con-

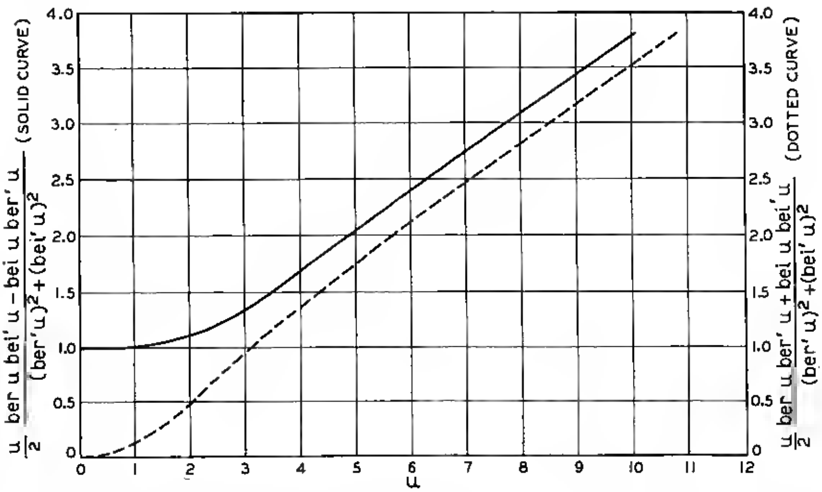


Fig. 2—The skin effect in solid wires. The upper curve represents the ratio of the a-c. resistance of the wire to its d-c. resistance and the lower curve the ratio of the internal reactance to the d-c. resistance.

ductor is thin, they are nearly equal at all frequencies. If the return path is partly internal and partly external, we have in effect two transmission lines with a distributed mutual impedance $Z_{ab}$ due to the mingling of the two currents in the hollow conductor common to both lines. However, since this quantity $Z_{ab}$ is not the total mutual impedance between the two lines unless the hollow conductor is the only part of the electromagnetic field common to them, it is better to call $Z_{ab}$ the *transfer impedance* from one surface of the conductor to the other.

In order to determine these impedances, let us suppose that of the total current $I_a + I_b$ flowing in the hollow conductor, the part $I_a$ returns inside and the rest outside. Since the total current enclosed by the inner surface of the given conductor is $-I_a$, and that enclosed by the outer surface is $I_b$, the magnetomotive intensity takes the values $-(I_a/2\pi a)$ and $(I_b/2\pi b)$, respectively, at these surfaces. This information is sufficient to determine the values of the constants $A$ and $B$ in the equation (59) governing current distribution. In fact, we

have

$$\begin{aligned} AI_1(\sigma a) + BK_1(\sigma a) &= -\frac{I_a}{2\pi a}, \\ AI_1(\sigma b) + BK_1(\sigma b) &= \frac{I_b}{2\pi b}, \end{aligned} \tag{71}$$

and therefore

$$\begin{aligned} A &= \frac{K_1(\sigma b)}{2\pi a D} I_a + \frac{I_1(\sigma a)}{2\pi b D} I_b, \\ B &= -\frac{I_1(\sigma b)}{2\pi a D} I_a - \frac{I_1(\sigma a)}{2\pi b D} I_b, \end{aligned} \tag{72}$$

where

$$D = I_1(\sigma b)K_1(\sigma a) - I_1(\sigma a)K_1(\sigma b). \tag{73}$$

Substituting these into the second equation of the set (59), we obtain the longitudinal electromotive intensity at any point of the conductor. We are interested, however, in its values at the surfaces since these values determine the surface impedances. Equating $\rho$ successively to $a$ and $b$, we obtain

$$\begin{aligned} E_z(a) &= Z_{aa}I_a + Z_{ab}I_b, \\ E_z(b) &= Z_{ba}I_a + Z_{bb}I_b, \end{aligned} \tag{74}$$

where [20]

$$\begin{aligned} Z_{aa} &= \frac{\eta}{2\pi a D} [I_0(\sigma a)K_1(\sigma b) + K_0(\sigma a)I_1(\sigma b)], \\ Z_{bb} &= \frac{\eta}{2\pi b D} [I_0(\sigma b)K_1(\sigma a) + K_0(\sigma b)I_1(\sigma a)], \\ Z_{ab} &= Z_{ba} = \frac{1}{2\pi g a b D}. \end{aligned} \tag{75}$$

The results embodied in equation (74) can be stated in the following two theorems:

*Theorem 1: If the return path is wholly external ($I_a = 0$) or wholly internal ($I_b = 0$), the longitudinal electromotive intensity on that surface of a hollow conductor which is nearest to the return path equals the corresponding surface impedance per unit length multiplied by the total current flowing in the conductor; and the intensity on the other surface equals the transfer impedance per unit length multiplied by the total current.*

[20] To obtain the last equation, it is necessary to use the identity

$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x}.$$

*Theorem 2: If the return path is partly external and partly internal the separate components of the intensity due to the two parts of the total current are calculated by the above theorem and then added to obtain the total intensities.*

At high frequencies, or when the conductors are very large, (75) can be replaced by much simpler approximate expressions.[21] If, however, we are compelled to use the rigorous equations in numerical computations, it is convenient to express the Bessel functions in terms of Thomson functions. Two of these, the ber and bei functions, or Thomson functions of the first kind, have already been introduced. The functions of the second kind are defined in an entirely analogous fashion as

$$K_0(x\sqrt{i}) = \mathrm{ker}\, x + i\, \mathrm{kei}\, x. \tag{76}$$

Differentiating, we have

$$\sqrt{i}\, K_0'(x\sqrt{i}) = -\sqrt{i}\, K_1(x\sqrt{i}) = \mathrm{ker}'\, x + i\, \mathrm{kei}'\, x, \tag{77}$$

so that

$$K_1(x\sqrt{i}) = -\frac{\mathrm{ker}'\, x + i\, \mathrm{kei}'\, x}{\sqrt{i}}. \tag{78}$$

All these subsidiary functions have been tabulated;[22] but the process of computing the impedances is laborious nevertheless.

## The Complex Poynting Vector[23]

In the preceding sections we have been able to determine the surface impedances of the coaxial conductors by reducing the field equations to the form of transmission line equations, and interpreting various terms accordingly. However, if the conductors are eccentric or of irregular shape, the *effective* surface impedances are more conveniently calculated by the use of the modified Poynting theorem.

This theorem states that, if $E$ and $H$ are the complex electromotive and magnetomotive intensities at any point, and if $E^*$ and $H^*$ are the conjugate complex numbers, then[24]

$$\iint [EH^*]dS = g\iiint (EE^*)dv + i\omega\mu \iiint (HH^*)dv. \tag{79}$$

[21] See portion of this text under the heading "Approximate Formulæ for the Surface Impedance of Tubular Conductors," page 557.

[22] British Association Tables, 1912, pp. 57–68; 1915, pp. 36–38; 1916, pp. 108–122.

[23] For an early application of the Complex Poynting vector see Abraham v. Föppl, Vol. 1 (Ch. 3, Sec. 3).

[24] The brackets signify the vector product and the parentheses the scalar product of the vectors so enclosed. The inward direction of the normal to the surface is chosen as the positive direction. The division by $4\pi$ does not occur if the consistent practical system of units is used as it is done in this paper.

To get an insight into the significance of this equation, let us consider a conductor which is part of a single-mesh circuit, and extend our integrals over the region occupied by this conductor. Then the first integral on the right of (79) represents twice the power dissipated in heat in the conductor, while $\mu\int\int\int(HH^*)dv$ is four times the average amount of magnetic energy stored in it.

On the other hand, when we look at the conductor from the standpoint of circuit theory, these two quantities are respectively $RI^2$ and $LI^2$; $R$ and $L$ being by definition the "resistance" and "inductance" of the conductor. Hence we have the equation,

$$\int\int [EH^*]_n dS = (R + i\omega L)I^2 = ZI^2, \tag{80}$$

from which the *impedance* $Z$ can be computed when the field intensities are known at the surface of the conductor.

If, on the other hand, the conductor is part of a two-mesh circuit and $I_1$ and $I_2$ are the amplitudes of the currents in meshes 1 and 2 respectively, the average amount of energy dissipated in heat per second can be regarded as made up of three parts, two of which are proportional to the squares of these amplitudes, while the third is proportional to their product. The first two of these parts being dependent on the magnitude of the current flowing in one mesh only are attributed to the *self-resistance* of the conductor to the corresponding current; the third part is attributed to the *mutual resistance* of the conductor. Designating the self-resistances by $R_{11}$ and $R_{22}$ and the mutual resistance by $R_{12}$, we represent the energy dissipated in heat in the form $1/2(R_{11}I_1^2 + 2R_{12}I_1I_2 + R_{22}I_2^2)$. Similarly, the average amount of energy stored in the conductor can be represented in the form $1/4(L_{11}I_1^2 + 2L_{12}I_1I_2 + L_{22}I_2^2)$, where $L_{11}$ and $L_{22}$ are called respectively *self-inductances* and $L_{12}$ *mutual inductance*. In this case, equation (79) can be written as follows:

$$\int\int [EH^*]_n dS = Z_{11}I_1^2 + 2Z_{12}I_1I_2 + Z_{22}I_2^2, \tag{81}$$

where the quantities $Z_{11}$, $Z_{22}$ and $Z_{12}$ are respectively the *self-impedances* and the *mutual impedance* of the conductor.

In general, if the conductor is part of a $k$-mesh circuit, we can obtain all its self-and mutual impedances by evaluating the integral $\int\int[EH^*]_n dS$ over its surface, and picking out the coefficients of various combinations of $I$'s.

We shall have an occasion to apply these results in computing the effect of eccentricity upon the resistance of parallel cylindrical conductors.

## Approximate Formulæ for the Surface Impedance of Tubular Conductors

The exact formulæ (75) for the internal impedances of a tubular conductor are hard to use for numerical computations, but simple approximations can be easily obtained if the modified Bessel functions are replaced by their asymptotic expansions and the necessary division performed as far as the second term. Thus, we have

$$
\begin{aligned}
Z_{bb} &= \frac{\eta}{2\pi b}\left[\coth \sigma t + \frac{\pi}{2\sigma}\left(\frac{3}{a}+\frac{1}{b}\right)\right],\\
Z_{aa} &= \frac{\eta}{2\pi a}\left[\coth \sigma t - \frac{\pi}{2\sigma}\left(\frac{3}{b}+\frac{1}{a}\right)\right],\\
Z_{ab} &= \frac{\eta}{2\pi\sqrt{ab}}\operatorname{csch} \sigma t,
\end{aligned}
\tag{82}
$$

where $t$ is the thickness of the tube. Separating the real and imaginary parts, we have

$$
\begin{aligned}
R_{bb} &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh u + \sin u}{\cosh u - \cos u} + \frac{a+3b}{16\pi g a b^2},\\
R_{aa} &= \frac{1}{2a}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh u + \sin u}{\cosh u - \cos u} - \frac{b+3a}{16\pi g b a^2},\\
R_{ab} &= \frac{1}{\sqrt{ab}}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh \frac{u}{2}\cos\frac{u}{2} + \cosh\frac{u}{2}\sin\frac{u}{2}}{\cosh u - \cos u},\\
\omega L_{bb} &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh u - \sin u}{\cosh u - \cos u},\\
\omega L_{aa} &= \frac{1}{2a}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh u - \sin u}{\cosh u - \cos u},\\
\omega L_{ab} &= \frac{1}{\sqrt{ab}}\sqrt{\frac{\mu f}{\pi g}}\frac{\sinh \frac{u}{2}\cos\frac{u}{2} - \cosh\frac{u}{2}\sin\frac{u}{2}}{\cosh u - \cos u}\\
|Z_{ab}| &= \frac{\sqrt{\mu f}}{\sqrt{\pi g a b}(\cosh u - \cos u)},
\end{aligned}
\tag{83}
$$

where $u = t\sqrt{2g\omega\mu}$.

It is obvious that in the equations for the self-resistances, the second terms represent the first corrections for curvature and vanish altogether if the conductors are plane. Although these formulæ were derived by using asymptotic expansions which are valid only when the argument is large, i.e., at high frequencies, the results are good even at low

frequencies, provided the tubular conductor is not too thick. Thus, if the frequency is 0, the first term in the above expression for $R_{bb}$ becomes $1/2\pi gbt$ which is the d.-c. resistance of the tube if its curvature is neglected. The second term only partially corrects for curvature, the error being of the order of $t^2/8b^2$. Hence, if the thickness of the tube is not more than 25 per cent of its *high-frequency* radius, that is, the radius of the surface nearest the return path, the error is less than 1 per cent. The formula for the mutual impedance is exceedingly good down to zero frequency for all ordinary thicknesses.

If the frequency is very high, further approximations can be made and the formulæ simplified as follows:

$$
\begin{aligned}
R_{bb} &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}} + \frac{a+3b}{16\pi gab^2}, \\
R_{aa} &= \frac{1}{2a}\sqrt{\frac{\mu f}{\pi g}} - \frac{b+3a}{16\pi gba^2}, \\
R_{ab} &= \frac{\sqrt{2}}{\sqrt{ab}}\sqrt{\frac{\mu f}{\pi g}}\, e^{-(u/2)} \cos\left(u - \frac{\pi}{4}\right), \\
\omega L_{bb} &= \frac{1}{2b}\sqrt{\frac{\mu f}{\pi g}}, \\
\omega L_{aa} &= \frac{1}{2a}\sqrt{\frac{\mu f}{\pi g}}, \\
\omega L_{ab} &= \frac{\sqrt{2}}{\sqrt{ab}}\sqrt{\frac{\mu f}{\pi g}}\, e^{-(u/2)} \cos\left(u + \frac{\pi}{4}\right), \\
|Z_{ab}| &= \frac{\sqrt{2}}{\sqrt{ab}}\sqrt{\frac{\mu f}{\pi g}}\, e^{-(u/2)}.
\end{aligned}
\tag{84}
$$

If the ratio of the diameters of the tube is not greater than 4/3, then we have the following formula for the surface transfer impedance:

$$\frac{|Z_{ab}|}{R_{\text{d.-c.}}} = \frac{u}{\sqrt{\cosh u - \cos u}}, \tag{85}$$

which is correct to within 1 per cent at any frequency. This ratio is illustrated in Fig. 3. The ratios of the mutual resistance and the mutual reactance to the d.-c. resistance are shown in Fig. 4.

In the case of self-resistances, we let

$$R_0 = \frac{1}{2\pi grt}, \tag{86}$$

where $r$ is the high frequency radius of the tube. Thus $R_0$ is the d.-c. resistance of the tube if the curvature is neglected. Then we have approximately

$$\frac{R}{R_0} = \frac{u}{2} \cdot \frac{\sinh u + \sin u}{\cosh u - \cos u} \pm \frac{t}{2r}, \tag{87}$$

if the tube is fairly thin. The curvature correction is positive if the

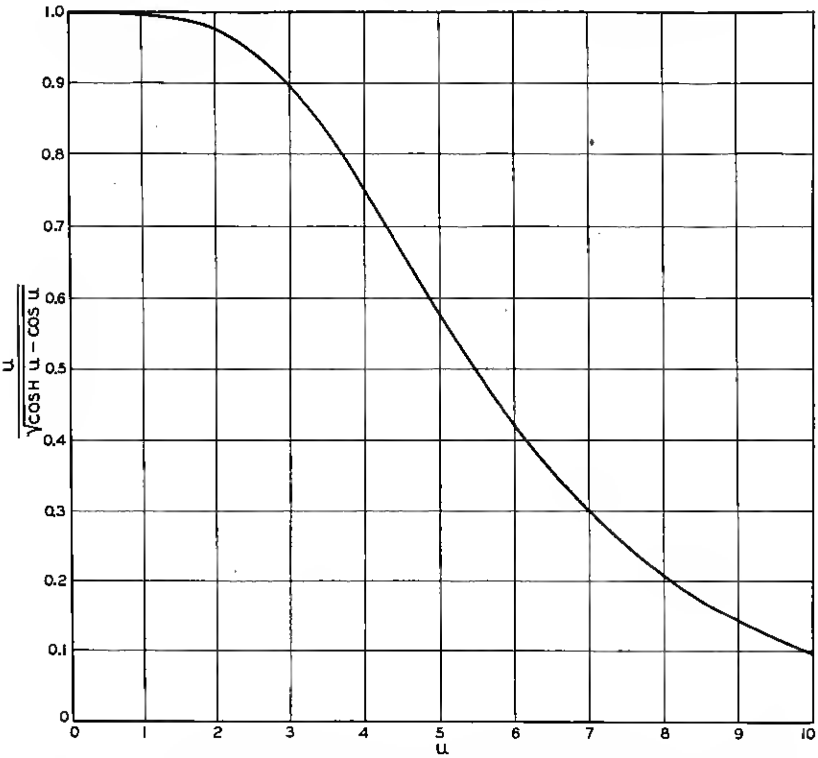


Fig. 3—The transfer impedance from one surface of a cylindrical shell to the other. The curve represents its ratio to the d-c. resistance.

return path is external, and negative if it is internal. The graph of the first term is shown in Fig. 5.

An interesting observation can be made at once from the formulæ (83) for the self-resistances of a tubular conductor. If the frequency is kept fixed and the thickness of the conductor is increased from 0, its resistance (with either return) passes through a sequence of maxima and minima.[25] The first minimum occurs when $u = \pi$, i.e., when

[25] The general fluctuating character of this function was noted by Mrs. S. P. Mead [12].

$t = \sqrt{\pi}/(2\sqrt{g\mu f})$; the first maximum occurs when $u = 2\pi$, etc. This fluctuation in resistance is due to the phase shift in the current density as we proceed from the surface of the conductor to deeper layers. The "optimum" resistance is $R_0((\pi/2) \tanh \pi/2) = 1.44R_0$, plus or

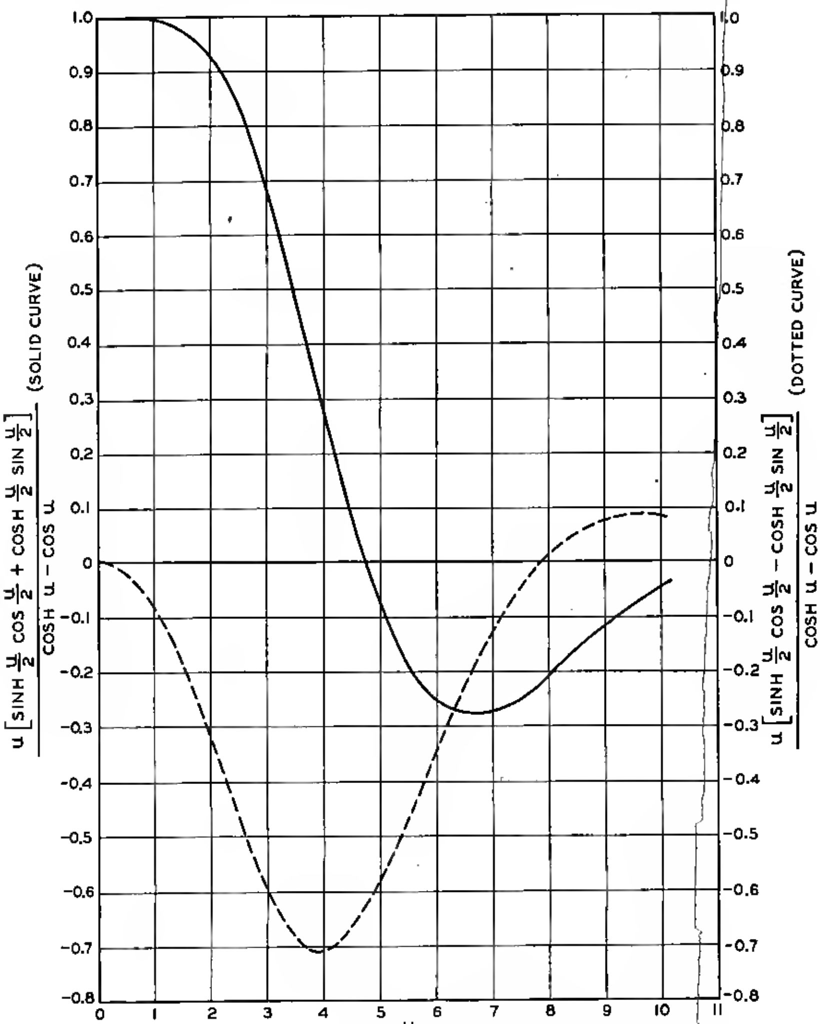


Fig. 4—The ratios of the transfer resistance and transfer reactance of a cylindrical shell to its d-c. resistance.

minus the curvature correction $t/2r$. If curvature is disregarded, the ratio of the optimum resistance to the resistance of the infinitely thick conductor with the same internal diameter as the hollow con-

ductor is $\tanh \pi/2 = 0.92$. When $u = 2\pi$, the ratio reaches its first maximum $\coth \pi = 1.004$. At 1 megacycle the optimum thickness of a copper conductor is about 0.1038 mm.

By a method of successive approximations, H. B. Dwight has obtained the impedance of a tubular conductor with an external coaxial

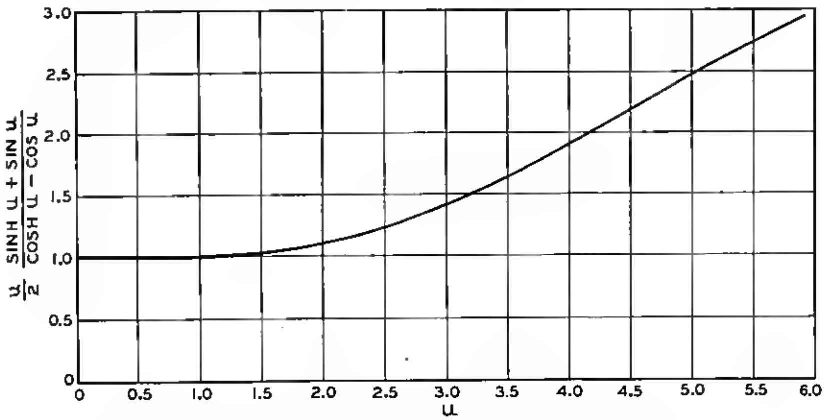


Fig. 5—The skin effect in cylindrical shells. The curve represents the ratio of the a-c. resistance of a typical shell to its d-c. resistance.

return.[26] His final results appear as the ratio of two infinite power series, which converge for all values of the variables involved, though they can be used advantageously in numerical computations only when the frequencies are fairly low and the convergence is rapid. We shall merely indicate how Dwight's formula and other similar formulæ can be obtained directly from the exact equations (75).

Let us replace the outer radius $b$ of (75) by $a + t$, where $t$ is the thickness of the wall, and replace the various Bessel functions by their Taylor series in $t$:

$$\begin{aligned} I_0(\sigma b) &= I_0(\sigma a + \sigma t) = \sum_{n=0}^{\infty} \frac{(\sigma t)^n}{n!} I_0^{(n)}(\sigma a), \\ K_0(\sigma b) &= K_0(\sigma a + \sigma t) = \sum_{n=0}^{\infty} \frac{(\sigma t)^n}{n!} K_0^{(n)}(\sigma a), \\ I_1(\sigma b) &= I_0'(\sigma b) = \sum_{n=0}^{\infty} \frac{(\sigma t)^n}{n!} I_0^{(n+1)}(\sigma a), \\ -K_1(\sigma b) &= K_0'(\sigma b) = \sum_{n=0}^{\infty} \frac{(\sigma t)^n}{n!} K_0^{(n+1)}(\sigma a). \end{aligned} \tag{88}$$

[26] "Skin Effect in Tubular and Flat Conductors," *A. I. E. E. Journal*, Vol. 37 (1918), p. 1379.

We thus obtain

$$Z_{bb} = \frac{2\eta}{b} \frac{\sum_{n=0}^{\infty} A_n \frac{(\sigma t)^n}{n!}}{\sum_{n=0}^{\infty} A_{n+1} \frac{(\sigma t)^n}{n!}}, \tag{89}$$

where $A_n$ is defined as

$$A_n = \begin{vmatrix} I_0'(\sigma a) & I_0^{(n)}(\sigma a) \\ K_0'(\sigma a) & K_0^{(n)}(\sigma a) \end{vmatrix}. \tag{90}$$

In spite of the complicated appearance of (90) the $A$'s are in reality very simple functions of $\sigma a$, as the accompanying list (91) will show.[27]

$$\begin{aligned} &A_0 = \frac{1}{\sigma a}, \qquad A_1 = 0, \qquad A_2 = \frac{1}{\sigma a}, \qquad A_3 = -\frac{1}{\sigma^2 a^2}, \\ &A_4 = \frac{1}{\sigma a} + \frac{3}{\sigma^3 a^3}, \qquad A_5 = -\frac{2}{\sigma^2 a^2} - \frac{12}{\sigma^4 a^4}, \\ &A_6 = \frac{1}{\sigma a} + \frac{9}{\sigma^3 a^3} + \frac{60}{\sigma^5 a^5}. \end{aligned} \tag{91}$$

The formula (90) can be made more rapidly convergent by partially summing the numerator and the denominator by means of hyperbolic functions. Thus, the numerator becomes

$$\sqrt{\frac{a}{b}} \cosh \sigma t + \frac{\sinh \sigma t}{2\sigma a}\left[1 - \frac{3t}{4a} + \cdots\right] - \frac{3t^2}{a^2} + \cdots,$$

and the denominator

$$\sqrt{\frac{a}{b}} \sinh \sigma t + \frac{3(\sigma t \cosh \sigma t - \sinh \sigma t)}{8(\sigma a)^2} + \cdots.$$

The reader will readily see that there would be no difficulty in using this method to obtain other expansions somewhat similar to (89). For example, we might write $a = b - t$ in (75) and express our results in terms of the outer radius. In this respect the method that we have used has greater flexibility than Dwight's; but there seems to be little advantage gained from it, since the simple formulæ (82) are sufficient for most practical purposes.

[27] The values given in (91) are *exact*, not approximate. One of them, namely,

$$A_1 = \begin{vmatrix} I_0'(\sigma a) & I_0(\sigma a) \\ K_0'(\sigma a) & K_0(\sigma a) \end{vmatrix} = \frac{1}{\sigma a},$$

is one of the fundamental identities found in all books on Bessel functions. The rest are consequences of analogous, though ess familiar, identities. The general expressions for the coefficients $A_n$ were obtained by H. Pleijel [20].

## Internal Impedances of Laminated Conductors

So far we have supposed that all conductors were homogeneous. We shall now consider a somewhat more general conductor composed of $n$ coaxial layers of different substances. As before, we are interested in finding expressions for the internal impedances; besides, we may wish to know how the total current is distributed between the different layers of the conductor.

To begin with, let us suppose that a coaxial return path is provided *outside* the given conductor. We number our layers consecutively and call the inner layer the first. Let $Z_{aa}^{(m)}$ and $Z_{bb}^{(m)}$ be the surface impedances of the $m$th layer, the first when the return is internal, the other when it is external; and let $Z_{ab}^{(m)}$ be the transfer impedance from one surface to the other. Formulæ for these impedances have already been obtained in the section under "The Surface Impedances of Hollow Cylindrical Shells," page 552. Also, let $z_{bb}^{(m)}$ be the *surface impedance of the first $m$ layers* with external return; that is, the ratio of the longitudinal electromotive intensity at the outer surface of the $m$th layer to the total current $I_m$ in all $m$ layers.[28]

By hypothesis, there is no return path inside the laminated conductor as a whole. Hence, when we fix our attention on any one layer alone, say the $m$th, we may say that the current *in this layer* returns partly through the $m - 1$ layers within it, and partly outside. In the $m - 1$ inner layers, however, the current is assumed to be $I_{m-1}$ in the outward direction—or what amounts to the same thing $-I_{m-1}$ in the return direction. Hence we conclude that, of the current $I_m - I_{m-1}$ in the layer under discussion, $I_m$ returns *outside* and $-I_{m-1}$ *inside*. Substituting these values in Theorem 2 on page 555, we find that the electromotive intensity along the inner surface of the layer is $Z_{ab}^{(m)}I_m - Z_{aa}^{(m)}I_{m-1}$.

But the *inner* surface of the $m$th layer is the *outer* surface of the composite conductor comprising the $m - 1$ inner layers, and by Theorem 1, the electromotive intensity on this outer surface is $z_{bb}^{(m-1)}I_{m-1}$. As the two must be equal, we obtain an equation from which we can determine the ratio of the current flowing in the first $m - 1$ layers to that flowing in $m$ layers. This is

$$\frac{I_{m-1}}{I_m} = \frac{Z_{ab}^{(m)}}{Z_{aa}^{(m)} + z_{bb}^{(m-1)}}. \tag{92}$$

In this formula for the effect of an extra layer on the current dis-

[28] In this notation, the current flowing in the $m$th layer is $I_m - I_{m-1}$. It should also be noted that $Z_{bb}^{(1)} = z_{bb}^{(1)}$.

tribution, it will be noted that the denominator is the impedance (with internal return) of the added layer plus the original impedance.

We now consider the electromotive intensity on the outer surface of the $m$th layer, which is $z_{bb}^{(m)}I_m$ on the one hand, and $(Z_{bb}^{(m)}I_m - Z_{ab}^{(m)}I_{m-1})$ on the other. Thus, we have the following equation,

$$z_{bb}^{(m)} = Z_{bb}^{(m)} - Z_{ab}^{(m)}\frac{I_{m-1}}{I_m} = Z_{bb}^{(m)} - \frac{[Z_{ab}^{(m)}]^2}{Z_{aa}^{(m)} + z_{bb}^{(m-1)}}, \tag{93}$$

expressing the effect of an additional layer upon the impedance of the conductor.

This equation is a convenient reduction formula. Starting with the first layer (for which $z_{bb}^{(1)} = Z_{bb}^{(1)}$), we add the remaining layers one by one and thus obtain the impedance of the complete conductor in the form of the following continued fraction:

$$z_{bb}^{(n)} = Z_{bb}^{(n)} - \frac{[Z_{ab}^{(n)}]^2}{Z_{aa}^{(n)} + Z_{bb}^{(n-1)} +} \frac{[Z_{ab}^{(n-1)}]^2}{Z_{aa}^{(n-1)} + Z_{bb}^{(n-2)} +} \cdots \frac{[Z_{ab}^{(2)}]^2}{Z_{aa}^{(2)} + Z_{bb}^{(1)}}. \tag{94}$$

We can also get a reduction formula for the transfer impedance between the inner and outer surfaces of the composite conductor formed by the first $m$ layers. To do so, it is only necessary to note that, since the inner surface of the first $m - 1$ layers is also the inner surface of the first $m$ layers as well, the electromotive intensity on that surface can be expressed either as $z_{ab}^{(m-1)}I_{m-1}$ or as $z_{ab}^{(m)}I_m$. Thus, we have

$$z_{ab}^{(m)} = z_{ab}^{(m-1)}\frac{I_{m-1}}{I_m} = \frac{Z_{ab}^{(m)}z_{ab}^{(m-1)}}{Z_{aa}^{(m)} + z_{bb}^{(m-1)}}. \tag{95}$$

By noting that $z_{ab}^{(1)} = Z_{ab}^{(1)}$, we can determine successively the transfer impedances across the first two layers, the first three, and so on. This formula is not quite as simple as (94), owing to the presence of $z_{bb}^{(m-1)}$ in its denominator, and it is therefore not expedient to evaluate $z_{ab}^{(m)}$ explicitly; but it is not prohibitively cumbersome from the numerical standpoint when the computations are made step by step.

Although in deducing equations (93) and (95) we supposed that the added layer was homogeneous, the equations are correct even if this layer consists of several coaxial layers, provided $Z_{aa}^{(m+1)}$ and $Z_{ab}^{(m+1)}$ are interpreted as the impedances of the added non-homogeneous layer in the absence of the original core of $m$ layers. These latter

impedances themselves have to be computed by means of equations (94) and (95).

If the return path is inside the laminated conductor, instead of outside, formulæ (92) and (93) still hold, provided we interchange $a$ and $b$, and count layers from the outside instead of the inside, so that $m = 1$ is the outermost, rather than the innermost, layer.

The basic rule for determining the surface impedances of laminated conductors can be put into the following verbal form:

*Theorem 3: Let two conductors, both of which may be made up of coaxial layers, fit tightly one inside the other. Any surface self-impedance of the compound conductor equals the individual impedance of the conductor nearest to the return path diminished by the fraction whose numerator is the square of the transfer impedance across this conductor and whose denominator is the sum of the surface impedances of the two component conductors if each is regarded as the return path for the other. The transfer impedance of the compound conductor is the fraction whose numerator is the product of the transfer impedances of the individual conductors and whose denominator is that of the self-impedance.*

If two coaxial conductors are short-circuited at intervals, short compared to the wave-length, the above theorem holds even if the conductors do not fit tightly one over the other, provided we add in the denominators a third term representing the inductive reactance of the space between the conductors.

## Disks as Terminal Impedances for Coaxial Pairs

So far we have been concerned only with infinitely long pairs. We now take up a problem of a different sort; namely, the design of a disk which, when clapped on the end of such a pair, will not give rise to a reflected wave.

The line of argument will be as follows: To begin with, we shall assume a disk of arbitrary thickness $h$, compute the field which will be set up in it, and then adjust the thickness so as to make this field match that which would exist in the dielectric of an infinite line.

The field in the disk has to satisfy equation (2) where $i\omega\epsilon$ can be disregarded by comparison with $g$. Thus, we have

$$\begin{gathered}\frac{\partial H_\varphi}{\partial z} = -\,gE_\rho, \qquad \frac{1}{\rho}\frac{(\rho H_\varphi)}{\partial \rho} = gE_z, \\ \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} = i\omega\mu H_\varphi.\end{gathered} \tag{96}$$

In the dielectric between the coaxial conductors, the longitudinal displacement current density is very small; in fact, it would be zero if the conductors were perfect. This current density is continuous across the surface of the disk and, therefore, $gE_z$ is exceedingly small. Hence, the second of the above equations becomes approximately

$$\frac{\partial(\rho H_\varphi)}{\partial \rho} = 0; \tag{97}$$

so that

$$H_\varphi = \frac{P}{\rho}, \tag{98}$$

where $P$ is independent of $\rho$ but may be a function of $z$. Under these conditions, the remaining two equations are

$$\frac{\partial E_\rho}{\partial z} = -i\omega\mu H_\varphi, \qquad \frac{\partial H_\varphi}{\partial z} = -gE_\rho. \tag{99}$$

From the form of these equations and from (98), we conclude that the general expressions for the intensities in the disk are

$$H_\varphi = \frac{Ae^{\sigma z} + Be^{-\sigma z}}{\rho}, \qquad E_\rho = \frac{\sigma[Be^{-\sigma z} - Ae^{\sigma z}]}{g\rho}, \tag{100}$$

where $\sigma = \sqrt{g\omega\mu i}$.

On the outside flat surface of the disk (given by $z = h$ where $h$ is the thickness of the plate), the magnetomotive intensity is very nearly zero;[29] therefore,

$$Ae^{\sigma h} + Be^{-\sigma h} = 0. \tag{101}$$

From this we obtain

$$A = -Ce^{-\sigma h}, \qquad B = Ce^{\sigma h}, \tag{102}$$

where $C$ is some constant. Thus equations (100) can be written as follows:

$$\begin{aligned} H_\varphi &= \frac{C \sinh \sigma(h - z)}{\rho}, \\ E_\rho &= \frac{\sigma C \cosh \sigma(h - z)}{g\rho}; \end{aligned} \tag{103}$$

and at the boundary between the disk and the dielectric of the transmission line ($z = 0$), we have

$$\frac{E_\rho}{H_\varphi} = \frac{\sigma}{g} \coth \sigma h. \tag{104}$$

[29] On account of the negligibly small longitudinal current in the disk.

On the other hand, if there is to be no reflection this must equal $\sqrt{\mu/\epsilon}$ by equation (24). Hence

$$\frac{\sigma}{g}\coth \sigma h = \sqrt{\frac{\mu}{\epsilon}}. \tag{105}$$

If $\sigma h$ is small, coth $\sigma h$ equals approximately $1/\sigma h$, and

$$h = \frac{1}{g}\sqrt{\frac{\epsilon}{\mu}}\text{ cm.} \tag{106}$$

Under these conditions, the generalized flux of energy across the inner surface of the disk is, in accordance with the text under "The Complex Poynting Vector," page 555, and equation (14),

$$\int_0^{2\pi}\int_{b'}^{a''} E_\rho H_\varphi{}^*\rho\, d\rho\, d\varphi = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\log\frac{a''}{b'}\, I^2. \tag{107}$$

Thus, the impedance of this disk is a pure resistance equal to the characteristic impedance of the coaxial pair.

## Cylindrical Waves and the Problem of Cylindrical Shields [30]

It is well known that when two transmission lines are side by side, to a greater or lesser extent they interfere with each other. This interference is usually analyzed into "electromagnetic crosstalk" and "electrostatic crosstalk."

Thus, electric currents in a pair of parallel wires produce a magnetic field with lines of force perpendicular to the wires. These lines cut the other pair of wires and induce in them electromotive forces, thereby producing what is usually called the "electromagnetic crosstalk"; this crosstalk is seen to be proportional to the current flowing in the first pair. The "electrostatic crosstalk," on the other hand, is caused by electric charges induced on the wires of the second system; these charges are proportional to the potential difference existing between the wires of the "disturbing" transmission line.

The distinction between two types of crosstalk is valid, although the terminology is somewhat unfortunate; the word "electromagnetic" is used in too narrow a sense and the word "electrostatic" is a contradiction in terms since electric currents and charges in a transmission line are variable. The terms "impedance crosstalk" and "admittance crosstalk" would be preferable because the former is due to a distributed mutual series impedance between two lines and the latter is produced by a distributed mutual shunt admittance.

[30] Since this paper was written, a related paper has been published by Louis V. King [18]. However, the physical picture here developed appears to be new. The earliest writer who treated the problem of electromagnetic shielding is H. Pleijel [21].

The crosstalk between two parallel pairs (this applies to twisted pairs as well) can be reduced by enclosing each pair in a cylindrical metallic shield. It is the object of this and the following two sections to develop a theory for the design of such shields.

This theory is based upon an assumption that in so far as the radial movement of energy toward and away from the wires is concerned we can disregard the non-uniform distribution of currents and charges along the length of the wires. No serious error is introduced thereby as long as the radius of the shield is small by comparison with the wave-length. The field around the wires is considered, therefore, as due to superposition of two two-dimensional fields of the types given by equations (4) and (5).

The actual computation of the effectiveness of a given shield will be reduced to an analogous problem in Transmission Line Theory.

Equations (4) and (5) are too general as they stand. Strictly speaking the effect of a shield upon an arbitrary two-dimensional field cannot be expressed by a single number. The field at various points outside the shield will be reduced by it in different ratios. However, any such field can be resolved into "cylindrical waves," each of which is reduced by the shield everywhere in the same ratio. Moreover, to all practical purposes the field produced by electric currents (or electric charges) in a pair of wires is just such a pure cylindrical wave.

Since both $E$ and $H$ are periodic functions of the coordinate $\varphi$, they can be resolved into Fourier series. The name "cylindrical waves" will be applied to the fields represented by the separate terms of the series. As the name indicates the wave fronts of these waves are cylindrical surfaces, although owing to relatively low frequencies and long wave-lengths used in practice the progressive motion of these waves is not clearly manifested except at great distances from the wires.

Turning our attention specifically to magnetic cylindrical waves *of the nth order*, and writing the field components tangential to the wave fronts in the form $E \cos n\varphi$ and $H \cos n\varphi$, we have from equations (5):

$$\frac{dE}{d\rho} = -i\omega\mu H, \quad \frac{d(\rho H)}{d\rho} = -\left[(g + i\omega\epsilon)\rho + \frac{n^2}{i\omega\mu\rho}\right]E. \quad (108)$$

From these we obtain

$$\rho^2 \frac{d^2E}{d\rho^2} + \rho\frac{dE}{d\rho} = [i\omega\mu(g + i\omega\epsilon)\rho^2 + n^2]E. \quad (109)$$

This equation, being of the second order, possesses two independent solutions: one for diverging cylindrical waves and the other for reflected waves. The ratio of $E$ to $H$ in the first case and its negative in the second will be called the *radial impedance* offered by the medium to cylindrical waves.

In the next section we shall determine radial impedances in dielectrics and metals and show that for all practical purposes the attenuation of cylindrical waves in metals is exponential. The significance of the radial impedance is the same as that of the characteristic impedance of a transmission line. When a cylindrical wave passes from one medium into another, a *reflection* takes place unless the radial impedances are the same in the two media. Thus if $E_0$ and $H_0$ are the *impressed* intensities (at the boundary between the two media), $E_r$ and $H_r$ the reflected and $E_t$ and $H_t$, the transmitted intensities, we have

$$E_0 + E_r = E_t \qquad \text{and} \qquad H_0 + H_r = H_t, \tag{110}$$

since both intensities must be continuous. On the other hand, if $k$ is the ratio of the impedance in the first medium to that in the second, then equations (110) become

$$kH_0 - kH_r = H_t \qquad \text{and} \qquad H_0 + H_r = H_t. \tag{111}$$

Solving we obtain

$$H_t = \frac{2k}{k+1} H_0 \qquad \text{and} \qquad E_t = \frac{2}{k+1} E_0. \tag{112}$$

The reflection loss will be defined as

$$R = 20 \log_{10} \frac{|H_0|}{|H_t|} \text{ decibels.} \tag{113}$$

When a wave passes through a shield, it encounters two boundaries and if the shield is electrically thick, that is, if the attenuation of the wave in the shield is so great that secondary reflections can be disregarded without introducing a serious error, the total reflection loss is the sum of the losses at each boundary. The first loss can be computed directly from (112) and the second from the same equation if we replace $k$ by its reciprocal. Thus, the total reflection loss for electrically thick shields is

$$R = 20 \log_{10} \frac{|k+1|^2}{4|k|} \text{ decibels.} \tag{114}$$

When the ratio of the impedances is very large by comparison with unity, the formula becomes

$$R = 20 \log_{10} \frac{|k|}{4}, \tag{115}$$

and when $k$ is very small, then

$$R = 20 \log_{10} \frac{1}{4|k|}. \tag{116}$$

In the next section we shall see that to all practical purposes, the wave in the shield is attenuated exponentially. If $\alpha$ is the attenuation constant in nepers and if $t$ is the thickness of the shield, then the attenuation loss is

$$A = 8.686\alpha t \text{ decibels} \tag{117}$$

and the total reduction in the magnetomotive intensity due to the presence of the shield is

$$S = R + A. \tag{118}$$

The electromotive intensity is reduced in the same ratio.

But if the shield is not electrically thick, a correction term has to be added to the reflection loss. This correction term can be shown to be [31]

$$C = 20 \log_{10} \left| 1 - \frac{(k-1)^2}{(k+1)^2} e^{-2\Gamma t} \right| \text{ decibels}, \tag{119}$$

and if $k$ is very large or very small by comparison with unity then

$$C \doteq 3 - 8.686\alpha t + 10 \log_{10} (\cosh 2\alpha t - \cos 2\beta t). \tag{120}$$

Equation (120) does not hold down to $t = 0$; when $\Gamma t$ is nearly zero, then

$$C = 20 \log_{10} \left| 1 - \frac{(k-1)^2}{(k+1)^2} \right|. \tag{121}$$

So far we supposed that the shields were coaxial with the source. If this is not so, it is always possible to replace any given line source within the shield by an equivalent system of line sources coaxial with the shield and emitting cylindrical waves of proper orders. Mathematically this amounts to a change of the origin of the coordinate system. In the next section we shall see that the shielding effectiveness is not the same for all cylindrical waves. This means, of course, that if the shield is not coaxial with the source, the total reduction in

[31] Here, $\Gamma = \alpha + i\beta$ is the propagation constant in the shield.

the field depends upon the position of the measuring apparatus. The variation is very small, however, unless the source is almost touching the shield and it can be stated that approximately the shielding effectiveness is independent of the position of the source.

It is interesting to observe from the accompanying tables that while the attenuation loss is greater in iron than in copper, the reflection loss is greater at a copper surface. In fact, at some frequencies the impedances of iron and air nearly match and practically no reflection takes place. Hence, a thin copper shield may be more effective than an equally thin iron shield. And if a composite shield is made of copper and iron, the shield will be more effective if copper layers are placed on the outside to take advantage of the added reflection.

TABLE I

THE ABSOLUTE VALUE OF THE RADIAL IMPEDANCE OFFERED BY AIR TO CYLINDRICAL MAGNETIC WAVES OF THE FIRST ORDER (IN MICROHMS)

| $f$ | Radius = 0.5 cm. | 1 cm. | 2 cm. |
|---|---|---|---|
| 1 cycle | 0.0395 | 0.07896 | 0.1579 |
| 10 cycles | 0.395 | 0.790 | 1.58 |
| 100 cycles | 3.95 | 7.90 | 15 8 |
| 1 kilocycle | 39.5 | 79.0 | 158. |
| 10 kilocycles | 395. | 790. | 1,580. |
| 100 kilocycles | 3,950. | 7,900. | 15,800. |
| 1 megacycle | 39,500. | 79,000. | 158,000. |
| 10 megacycles | 395,500. | 790,000. | 1.58 ohms |
| 100 megacycles | 3.95 ohms | 7.9 ohms | 15.8 ohms |

TABLE II

THE INTRINSIC IMPEDANCE OF CERTAIN METALS $(\eta)/(\sqrt{i})$ IN MICROHMS

| $f$ | Copper $g = 5.8005 \times 10^5$ mhos/cm. $\mu = 0.01257\ \mu h$/cm. | Lead $g = 4.8077 \times 10^4$ mhos/cm. $\mu = 0.01257\ \mu h$/cm. | Aluminum $g = \frac{1}{3}10^6$ mhos/cm. $\mu = 0.01257\ \mu h$/cm. | Iron $g = 10^5$ mhos/cm. $\mu = 1.257\ \mu h$/cm. = (100 relative to copper) |
|---|---|---|---|---|
| 1 cycle | 0.369 | 1.28 | 0.487 | 8.88 |
| 10 cycles | 1.17 | 4.05 | 1.54 | 28.1 |
| 100 cycles | 3.69 | 12.8 | 4.87 | 88.8 |
| 1 kilocycle | 11.7 | 40.5 | 15.4 | 281. |
| 10 kilocycles | 36.9 | 128. | 48.7 | 888. |
| 100 kilocycles | 117. | 405. | 154. | 2,810. |
| 1 megacycle | 369. | 1,280. | 487. | 8,880. |
| 10 megacycles | 1,170. | 4,050. | 1,540. | 28,100. |
| 100 megacycles | 3,690. | 12,800. | 4,870. | 88,800. |

TABLE III

The Intrinsic Propagation Constant of Certain Metals

| $f$ | Copper $g = 5.8005 \times 10^5$ mhos cm. $\mu = 0.01257$ $\mu$h/cm. | | Lead $g = 4.8077 \times 10^4$ mhos/cm. $\mu = 0.01257$ $\mu$h/cm. | | Aluminum $g = \frac{1}{3}10^6$ mhos/cm. $\mu = 0.01257$ $\mu$h/cm. | | Iron $g = 10^5$ mhos/cm. $\mu = 1.257$ $\mu$h/cm. = (100 relative to copper) | |
|---|---|---|---|---|---|---|---|---|
| | $\frac{\sigma}{\sqrt{i}}$ in $\frac{\text{nepers}}{\text{cm.}}$ | $\frac{\sigma}{\sqrt{i}}$ in db/cm. | $\frac{\sigma}{\sqrt{i}}$ in $\frac{\text{nepers}}{\text{cm.}}$ | $\frac{\sigma}{\sqrt{i}}$ in db/cm. | $\frac{\sigma}{\sqrt{i}}$ in $\frac{\text{nepers}}{\text{cm.}}$ | $\frac{\sigma}{\sqrt{i}}$ in db/cm. | $\frac{\sigma}{\sqrt{i}}$ in $\frac{\text{nepers}}{\text{cm.}}$ | $\frac{\sigma}{\sqrt{i}}$ in db/cm. |
| 1 cycle | 0.214 | 1.86 | 0.0616 | .535 | 0.162 | 1.41 | 0.888 | 7.72 |
| 10 cycles | 0.677 | 5.88 | 0.195 | 1.69 | 0.513 | 4.46 | 2.81 | 24.4 |
| 100 cycles | 2.14 | 18.6 | 0.616 | 5.35 | 1.62 | 14.1 | 8.88 | 77.2 |
| 1 kilocycle | 6.77 | 58.8 | 1.95 | 16.9 | 5.13 | 44.6 | 28.1 | 244. |
| 10 kilocycles | 21.4 | 186. | 6.16 | 53.5 | 16.2 | 141. | 88.8 | 772. |
| 100 kilocycles | 67.7 | 588. | 19.5 | 169. | 51.3 | 446. | 281. | 2,440. |
| 1 megacycle | 214. | 1,860. | 61.6 | 535. | 162. | 1,410. | 888. | 7,720. |
| 10 megacycles | 677. | 5,880. | 195. | 1690. | 513. | 4,460. | 2810. | 24,400. |
| 100 megacycles | 2140. | 18,600. | 616. | 5350. | 1620. | 14,100. | 8880. | 77,200. |

## Cylindrical Waves in Dielectrics and Metals

In good dielectrics $g$ is small by comparison with $\omega\epsilon$ and the first term on the right in (109) very nearly equals $(2\pi\rho/\lambda)^2$ where $\lambda$ is the wave-length. But we are interested in wave-lengths measured in miles and shields with diameters measured in inches; thus we shall write (109) in the following approximate form:

$$\rho^2 \frac{d^2E}{d\rho^2} + \rho \frac{dE}{d\rho} - n^2E = 0. \tag{122}$$

When $n \neq 0$ there are two independent solutions

$$E_1 = \rho^{-n} \quad \text{and} \quad E_2 = \rho^n; \tag{123}$$

and when $n = 0$,

$$E_1 = \log \rho \quad \text{and} \quad E_2 = 1. \tag{124}$$

The corresponding expressions for $H$ are, by (108),

$$H_1 = \frac{n\rho^{-n-1}}{i\omega\mu} \quad \text{and} \quad H_2 = -\frac{n\rho^{n-1}}{i\omega\mu}, \tag{125}$$

in the first case, and

$$H_1 = \frac{1}{i\omega\mu\rho} \quad \text{and} \quad H_2 = 0, \tag{126}$$

in the second.

The second case in which $E_1$ and $H_1$ are the electromotive and magnetomotive intensities in the neighborhood of an isolated wire carrying electric current is of interest to us only in so far as it helps to interpret (123) and (125). If we were to consider $2n$ infinitesimally thin wires equidistributed upon the surface of an infinitely narrow cylinder, the adjacent wires carrying equal but oppositely directed currents of strength sufficient to make the field different from zero, and calculate the field, we should obtain expressions proportional to $E_1$ and $H_1$. An actual cluster of $2n$ wires close together would generate principally a cylindrical wave of order $n$; the strengths of other component waves of order $3n$, $5n$, etc. rapidly diminish as the distance from the cluster becomes large by comparison with the distance between the adjacent wires of the cluster. For the purposes of shielding design we can regard a pair of wires as generating a cylindrical wave of the first order ($n = 1$). The radial impedance of an $n$th order wave is

$$Z_\rho = \frac{E_1}{H_1} = \frac{i\omega\mu\rho}{n}, \tag{127}$$

and that of the corresponding reflected wave has the same value. It should be noted that by the "reflected" cylindrical wave in the space enclosed by a shield, we mean the sum total of an infinite number of successive reflections. Each of the latter waves condenses on the axis and diverges again only to be re-reflected back; in a steady state all these reflected waves interfere with each other and form what might be called a "stationary reflected wave." Not being interested in any other kind of reflected waves we took the liberty of omitting the qualification.

In conductors the attenuation of a wave due to energy dissipation is much greater (except at extremely low frequencies) than that due to the cylindrical divergence of the wave. Hence, in the shield we can regard the wave as plane and write (108) in the following approximate form:

$$\frac{dE}{d\rho} = -\,i\omega\mu H, \qquad \frac{dH}{d\rho} = -\,gE. \tag{128}$$

In form, these are exactly like ordinary transmission line equations. Hence, in a shield the radial impedance is simply the intrinsic impedance of the metal,

$$Z_\rho = \eta = \sqrt{\frac{i\omega\mu}{g}}\ \text{ohms}, \tag{129}$$

and the propagation constant,

$$\sigma = \sqrt{i\omega\mu g} = \sqrt{\pi f\mu g}\,(1 + i)\ \text{nepers/cm}. \tag{130}$$

The exact value of the radial impedance in metals can be found by solving (108). Thus, we can obtain

$$Z_\rho = -\,\eta\frac{K_n(\sigma\rho)}{K_n'(\sigma\rho)} \tag{131}$$

for diverging waves, and

$$Z_\rho = \eta\frac{I_n(\sigma\rho)}{I_n'(\sigma\rho)} \tag{132}$$

for the reflected waves.

Cylindrical waves of the electric type can be treated in the same manner. It turns out that the transmission laws in metals are identical with those for magnetic waves. The radial impedance in perfect dielectrics, on the other hand is given by

$$Z_\rho = \frac{n}{i\omega\epsilon\rho}\,. \tag{133}$$

This is enormous by comparison with the impedance in metals, thereby explaining an almost perfect "electrostatic" shielding offered by metallic substances. Even when the frequency is as high as 100 kc. the radial impedance of air 1 cm. from the source is about $36 \times 10^6$ ohms while the impedance of a copper shield is only $117 \times 10^{-6}$ ohms. The reflection loss is approximately 220 db.

## Power Losses in Shields

As we have shown in the text under "The Complex Poynting Vector," page 555, the average power dissipated in a conductor is the real part of the integral $\Phi = 1/2\int\int[EH^*]_n dS$ taken over the surface of the conductor. If the source of energy is inside a shield, the integration need be extended only over its inner surface, because the average energy flowing outward through this surface is almost entirely dissipated in the shield, the radiation loss being altogether negligible. If a cylindrical wave whose intensities at the inner surface of the shield of radius "$a$" are

$$H_\varphi = H_0 \cos n\varphi, \qquad H_\rho = H_0 \sin n\varphi, \qquad E_z = \eta H_\varphi, \tag{134}$$

$\eta = i\omega\mu a/n$ being the radial impedance in the dielectric, is impressed upon the inner surface of the shield, a reflected wave is set up. The resultant of the magnetomotive intensities in the two is readily found to be $(2k/k + 1)H_0$, where $k$ is the ratio of the radial impedance of the dielectric column inside the shield to the impedance $Z$ looking into the shield. If the shield is electrically thick, the impedance $Z$ is obviously the radial impedance of the shield; otherwise it is modified somewhat by reflection from the outside of the shield. The average power loss in the shield per centimeter of length is, then, the real part of

$$\Phi = \frac{2\pi akk^*Z}{(k+1)(k^*+1)} H_0 H_0^*. \tag{135}$$

This becomes simply

$$\Phi = 2\pi a Z H_0 H_0^*, \tag{136}$$

if the frequency is so high that $k$ is large as compared with unity.

If the source of the impressed field is a pair of wires along the axis of the shield, the magnetomotive intensity on the surface of the shield can be shown to be

$$H_\varphi = \frac{l}{2\pi a^2} I \cos \varphi, \tag{137}$$

where $l$ is the separation between the axes of the wires. Therefore,

$$\Phi = \frac{kk^*l^2Z}{2\pi a^3(k+1)(k^*+1)} I^2. \tag{138}$$

## Resistance of Nearly Coaxial Tubular Conductors

When two tubular conductors are not quite coaxial, a proximity effect [32] appears which disturbs the symmetry of current distribution and therefore somewhat increases their resistance. This effect can be estimated by the following method of successive approximations. To begin with, we assume a symmetrical current distribution in the *inner* conductor. The magnetic field outside this conductor is then the same as that of a simple source along its axis and can be replaced by an equivalent distribution of sources situated along the axis of the outer conductor. The principal component of this distribution is a simple source of the same strength as the actual source and does not enter into the proximity effect. The next largest component is a double source given by

$$E_z = \frac{i\omega\mu lI}{2\pi r}\cos\theta, \qquad H_\theta = \frac{lI}{2\pi r^2}\cos\theta, \tag{139}$$

where $l$ is the interaxial separation, $r$ is the distance of a typical point of the field from the axis of the outer conductor, and $\theta$ is the remaining polar coordinate.

This field is impressed upon the inner surface of the outer conductor [33] and the resulting power loss equals, by equation (136), the real part of

$$\Phi = 2\pi a\eta\left(\frac{l}{2\pi a^2}\right)^2 I^2 = \frac{l^2}{2\pi a^3}\eta I^2, \tag{140}$$

where at high frequencies $\eta = \sqrt{i\omega\mu/g}$ is simply the intrinsic impedance of the outer conductor.[34] This loss increases the resistance of the outer tube by the amount,

$$\Delta R_a = \frac{l^2}{a^3}\sqrt{\frac{\mu f}{\pi g}}, \tag{141}$$

[32] For promixity effect in parallel wires external to each other, the reader is referred to the following papers: John R. Carson [1], C. Manneback [9], S. P. Mead [12].

[33] The radius of this surface is designated by $a$.

[34] At low frequencies $\eta$ has to be replaced by the radial impedance looking into the shield.

in excess of the concentric resistance $R_a = (1/2a)\sqrt{\mu f/\pi g}$ given by (84). The relative increase is, therefore,

$$\frac{\Delta R_a}{R_a} = \frac{2l^2}{a^2}. \tag{142}$$

The magnetic field (139) is partially reflected from the outer tube, impressed upon the inner conductor, partially refracted into it and dissipated there. Using (110) and (111) we can show that the reflected field is

$$\begin{aligned} H_\theta &= \frac{lI}{2\pi a^2}\cos\theta, \\ E_z &= \frac{i\omega\mu lI}{2\pi a^2}\rho\cos\theta. \end{aligned} \tag{143}$$

This field converges to the axis of the outer conductor. In order to estimate its effect upon the inner conductor, it is convenient to replace it by an equivalent field converging toward the axis of the inner conductor. By properly changing the origin of the coordinate system this equivalent field can be shown to be

$$\begin{aligned} E_z &= -\frac{i\omega\mu lI}{2\pi a^2}(l + \rho\cos\varphi), \\ H_\varphi &= -\frac{lI}{2\pi a^2}\cos\varphi. \end{aligned} \tag{144}$$

Applying once more (138) (replacing there $a$ by the radius $b$ of the inner conductor), we find that the power loss due to this field is given by the real part of

$$\Phi = \frac{bl^2}{2\pi a^4}\ I^2, \tag{145}$$

so that the absolute increase in resistance of the inner conductor is

$$\Delta R_b = \frac{bl^2}{a^4}\sqrt{\frac{\mu f}{\pi g}} \tag{146}$$

which must be added to the concentric resistance of the inner conductor $R_b = (1/2b)\sqrt{\mu f/\pi g}$. The relative increase is therefore

$$\frac{\Delta R_b}{R_b} = 2\left(\frac{b}{a}\right)^2\left(\frac{l}{a}\right)^2. \tag{147}$$

It is unnecessary to carry the process further.

Considering the pair as a whole, the resistance when concentric is $R = R_a + R_b$, and the increase due to eccentricity is $\Delta R = \Delta R_a + \Delta R_b$ thus giving a percentage increase,

$$\frac{\Delta R}{R} = 2\left(\frac{b}{a}\right)\left(\frac{l}{a}\right)^2. \tag{148}$$

It is obvious that, so long as $b$ and $l$ are small compared with $a$, this percentage increase is very small.

From the well-known formulæ for the inductance and the capacity between parallel cylindrical conductors, we find that the characteristic impedance of a nearly coaxial pair is given in terms of the characteristic impedance of the coaxial pair by

$$Z = Z_0\left[1 - \frac{e^2k^2}{(k^2 - 1)\log k}\right], \tag{149}$$

where the "eccentricity" $e$ is defined as the ratio of the interaxial separation to the inner radius of the outer conductor and $k$ as the ratio of the inner radius of the outer conductor to the outer radius of the inner conductor. Combining (149) and (148) we have for the attenuation of the nearly coaxial pair:

$$\alpha = \alpha_0\left[1 + \frac{2e^2}{k} + \frac{e^2k^2}{(k^2 - 1)\log k}\right]. \tag{150}$$

## References

### *Papers*


1. John R. Carson, "Wave Propagation Over Parallel Wires: The Proximity Effect," *Phil. Mag.*, Vol. 41, Series 6, pp. 607–633, April, 1921.
2. John R. Carson and J. J. Gilbert, "Transmission Characteristics of the Submarine Cable," *Jour. Franklin Institute*, p. 705, December, 1921.
3. John R. Carson and J. J. Gilbert, "Transmission Characteristics of the Submarine Cable," *Bell Sys. Tech. Jour.*, pp. 88–115, July, 1922.
4. John R. Carson, "The Guided and Radiated Energy in Wire Transmission," *A.I.E.E. Jour.*, pp. 908–913, October, 1924.
5. John R. Carson, "Electromagnetic Theory and the Foundations of the Electric Circuit Theory," *Bell Sys. Tech. Jour.*, January, 1927.
6. S. Butterworth, "Eddy Current Losses in Cylindrical Conductors, with Special Applications to the Alternating Current Resistances of Short Coils," *Phil. Trans., Royal Soc. of London*, pp. 57–100, September, 1921.
7. H. B. Dwight, "Skin Effect and Proximity Effect in Tubular Conductors," *A.I.E.E. Jour.*, Vol. 41, pp. 203–209, March, 1922.
8. H. B. Dwight, "Skin Effect and Proximity Effect in Tubular Conductors," *A.I.E.E. Trans.*, Vol. 41, pp. 189–195, 1922.
9. C. Manneback, "An Integral Equation for Skin Effect in Parallel Conductors," *Jour. of Math. and Physics*, April, 1922.
10. H. B. Dwight, "A Precise Method of Calculation of Skin Effect in Isolated Tubes," *A.I.E.E. Jour.*, Vol. 42, pp. 827–831, August, 1923.
11. H. B. Dwight, "Proximity Effect in Wires and Thin Tubes," *A.I.E.E. Jour.*, Vol. 42, pp. 961–970, September, 1923; *Trans.*, Vol. 42, pp. 850–859, 1923.

12. Mrs. S. P. Mead, "Wave Propagation Over Parallel Tubular Conductors: The Alternating Current Resistance," *Bell Sys. Tech. Jour.*, pp. 327–338, April, 1925.
13. Chester Snow, "Alternating Current Distribution in Cylindrical Conductors," *Scientific Papers of the Bureau of Standards*, No. 509, 1925.
14. John R. Carson and Ray S. Hoyt, "Propagation of Periodic Currents over a System of Parallel Wires," *Bell Sys. Tech. Jour.*, pp. 495–545, July, 1927.
15. John R. Carson, "Rigorous and Approximate Theories of Electrical Transmission Along Wires," *Bell Sys. Tech. Jour.*, January, 1928.
16. John R. Carson, "Wire Transmission Theory," *Bell Sys. Tech. Jour.*, April, 1928.
17. A. Ermolaev, "Die Untersuchung des Skineffektes—Drahten mit Complexer Magnetischer Permeabilitate," *Archiv. f. Elektrotechnik*, Vol. 23, pp. 101–108, 1929.
18. Louis V. King, "Electromagnetic Shielding at Radio Frequencies," *Phil. Mag.*, Vol. 15, Series 7, pp. 201–223, February, 1933.
19. E. J. Sterba and C. B. Feldman, "Transmission Lines for Short-Wave Radio Systems," *Proc. I. R. E.*, July, 1932, and *Bell Sys. Tech. Jour.*, July, 1932.
20. H. Pleijel, "Beräkning af Motstånd och Självinduktion," Stockholm, K. L. Beckmans Boktryckeri, 1906.
21. H. Pleijel, "Electric and Magnetic Induction Disturbances in Parallel Conducting Systems," 1926, *Ingeniorsvetenskapsakademiens Handlingar* NR 49.
22. J. Fisher, "Die allseitige in zwei Kreiszylindrishen, konaxial geschichteten Stoffen bei axialer Richtung des Wechselstromes," Jahrbuch der drahtlosen Telegraphie und Telephonie, Band 40, 1932, pp. 207–214.

*Books*

J. Clerk Maxwell, "Electricity and Magnetism," Vols. 1 and 2.
O. Heaviside, "Electrical Papers."
Sir William Thomson, "Mathematical and Physical Papers."
Lord Rayleigh, "Scientific Papers."
Sir J. J. Thomson, "Recent Researches in Electricity and Magnetism."
A. Russell, "A Treatise on the Theory of Alternating Currents."
John R. Carson, "Electric Circuit Theory and the Operational Calculus."
R. W. Pohl, "Physical Principles of Electricity and Magnetism."
Max Abraham and R. Becker, "The Classical Theory of Electricity and Magnetism."
E. Jahnke and F. Emde, "Tables of Functions," B. G. Teubner, 1933.

Note: This list of references is by no means complete. Only the more recent papers dealing with some phase of the subject treated here are included.